

Coarse Competitive Equilibrium and Extreme Prices^{*}

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Abstract

We introduce a notion of *coarse competitive equilibrium* (CCE), to study agents' inability to tailor their consumption to the state of the economy. Our notion is motivated by limited cognitive ability (in particular attention, memory, and complexity) and it maintains the complete market structure of competitive equilibrium. Compared to standard competitive equilibrium, our concept yields riskier allocations and more extreme prices. We provide a tractable model that is suitable for general equilibrium analysis as well as asset pricing.

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1 Introduction

Standard consumer theory assumes that agents adjust consumption in response to every change in price. In this paper, we modify this assumption and assume the agent's response is *coarse*, that is, she forms k categories and adjusts her consumption only if the price falls into a different category. The consumer forms her price categories optimally, that is, she chooses them to maximize her ex ante utility.

Consider, for example, an agent who forms two categories. This agent partitions circumstances into *high consumption states* and *low consumption states* and chooses one consumption level for each category. Thus, the agent makes two decisions: how to define the two categories *and* how much to consume in each. While the second decision is standard, the first one is not—this is our device for modeling how limited information processing or attention allocation responds to economic incentives. Note that in a stylized model such as ours the mathematical formulation of the optimization problem may be more complicated than the standard decision problem. Optimally partitioning the state space may seem like a difficult task. However, our model is not meant as a description of an agent's reasoning process. Rather, we aim to model an agent who, while unable to react to all price changes, responds to incentives when allocating cognitive resources.

To simplify the exposition, we assume the economy has two periods and that there is a single physical good.¹ The two-period model has a planning period and a consumption period. Households learn the state after the planning period and before the consumption period. In the planning period, each agent chooses an optimal categorization and an optimal consumption plan for those categories. We assume that agents are expected utility maximizers with a CRRA utility index. We show that under those assumptions the optimal categorization takes the following simple form: the agent identifies k contiguous price intervals and consumes the same amount for each price in a given interval.

The main focus of this paper is on general equilibrium implications of our model of consumer behavior. We analyze an endowment economy with a continuum of agents, each subject to the the above described coarseness constraint. We refer to the resulting equilibrium as a

¹We extend the model to an infinite horizon in the online appendix. We briefly discuss this extension in Section 5.5.

coarse competitive equilibrium (CCE). We show that a CCE exists and is Pareto optimal (given the restriction to coarse consumption plans). Notice that a single coarse consumption plan cannot distinguish among all states. Therefore, if all consumers classified states into the same categories, markets would not clear. Hence, in a CCE, ex ante identical agents may and sometimes must choose distinct plans. An implication of this observation is that consumption in a CCE is more risky than consumption in a competitive equilibrium with standard consumers.

To study CCE prices in an economy with a non-atomic endowment distribution, we fix the agents' utility function and consider a sequence of discrete endowment economies. We show that, in the non-atomic limit, CCE prices in states near the lowest or the highest possible endowment realization are *extreme*; the price of consumption diverges when the endowment is at or near the lower bound of the distribution while the price converges to zero when the endowment is near or at the upper bound of the distribution.

The intuition behind our extreme-price results is as follows. Equilibrium prices decrease monotonically in the aggregate endowment and each agent optimally partitions prices into k intervals and consumes the same amount for all prices in a given interval. In an optimal categorization, the agent does not want to “waste” a category for an event that is very unlikely unless identifying this event is very profitable. Market clearing and aggregate endowment variability imply that at least some agents must pay attention the upper tail of the distribution, while other agents must focus on the lower tail. In each of these cases, the agent uses up a valuable category for a very unlikely event. To make this choice of categories optimal, prices must be such that if the unlikely event occurs, those agents who pay attention to it get a large benefit. Households who identify endowments near the lower bound sell their endowment and consume little; therefore, for this strategy to be beneficial, prices at the lower bound must be much higher than in an economy without the coarseness constraint. Households who identify endowments near the upper bound benefit by consuming large amounts and, therefore, prices must be lower than in an economy without the coarseness constraint.

Our analysis highlights a particular mechanism behind extreme prices: consumers allocate their cognitive resources by responding to market incentives and price volatility serves to attract their attention. Though our analysis is performed in the context of a specific model, the basic message is robust to various modeling assumptions. In the last section of the paper we examine

how our results would change if we allowed agents to be differentiated by their risk posture and their complexity constraint; if instead of coarse consumption we assumed a coarseness constraint on net trades; and, to what extent our conclusions depend on the assumption of constant relative risk aversion.

1.1 Relation to Literature

The game theory literature has developed strategic analogs of coarse equilibrium. [Neyman \(1985\)](#), [Rubinstein \(1986\)](#), and [Abreu and Rubinstein \(1988\)](#) limit players' strategies in a repeated game to those implementable by finite state automata. Our approach is closest to [Neyman \(1985\)](#) who studies Nash equilibria of a game in which the number of states in the automaton is bounded. [Abreu and Rubinstein \(1988\)](#) also study Nash equilibria, but with a different cost function. [Rubinstein \(1986\)](#) examines a lexicographic cost of complexity and imposes a version of subgame perfection which precludes agents from adopting a different automaton later in the game. [Jehiel \(2005\)](#) and [Jehiel and Samet \(2007\)](#) constrain players to respond identically in similar situations by bundling their decision nodes into exogenous analogy classes. [Mengel \(2012\)](#) studies the evolutionary dynamics of categorization where the optimal partition size is determined in equilibrium, assuming a fixed marginal cost per partition cell.

The idea of a decision maker with a coarse understanding of the state space appears also in the literature on individual decision making, such as [Ahn and Ergin \(2010\)](#), [Dekel, Lipman, and Rustichini \(2001\)](#), and [Epstein, Marinacci, and Seo \(2007\)](#). In [Masatlioglu, Nakajima, and Ozbay \(2012\)](#) agents make optimal choices subject to an endogenous attention constraint. In [Dow \(1991\)](#)'s model of search with limited memory, the agent optimally partitions histories. [Piccione and Rubinstein \(1997\)](#) examine the relation between limited memory (i.e., imperfect recall) and time consistency. [Fryer and Jackson \(2008\)](#) show how optimal categorization can lead to such biases as discrimination against minorities based on statistical grounds. Likewise, [Wilson \(2002\)](#) analyzes long-run inference and shows that the optimal use of a limited memory can lead to many well-studied behavioral biases. [Mohlin \(2014\)](#) studies optimal categorization in prediction tasks; in his model the cost and benefits a partition are determined by a bias-variance tradeoff. [Mullainathan \(2002\)](#) studies a model of coarse categorization and its implications for asset returns and their trade volume.

Coarse understanding is also at the heart of the recent literature on rational inattention, which focuses on how information processing frictions impact asset prices and responses to monetary policy. [Sims \(2003\)](#) assumes that agents allocate their attention optimally subject to an information-theoretic constraint, allowing for information structures with noise. We focus on partitional information structures and limit the number of possible signal values instead of using the entropy based constraint. [Woodford \(2012\)](#) modifies Sims’ cost function to address consumer choice anomalies; the restriction of his model to partitional information leads to a similar constraint as ours. [Ellis \(2015\)](#) studies general cost functions for partitional information structures; his constrained attention model provides an axiomatic foundation for our work. [Mankiw and Reis \(2002\)](#) study a model that has only a fraction of agents getting new information each period. [Gabaix \(2014\)](#) offers a very different model leading to volatile prices: he solves a quadratic approximation to the optimal attention problem, adapted from the lasso method in statistics ([Tibshirani, 1996](#)).

Despite the differences in modeling details, all these papers, including ours, restrict agents’ ability to tailor their behavior to their environment by imposing additional constraints. Our main goal is to develop a tractable competitive equilibrium model with this type of constraint and analyze its effect on equilibrium prices.

2 Coarse Consumers

Consumers in our model choose consumption plans with limited complexity. Let $N = \{1, \dots, n\}$ be a finite set of states of the world. A partition $S = \{S_1, \dots, S_k\}$ of N represents a consumer’s choice as to what to pay attention to. Specifically, if states i and j are in the same partition element S_l , then the agent must choose the same consumption in both states. We measure complexity by the number of cells in the partition and assume that all agents choose partitions $S \in \mathcal{P}(k)$ with at most k cells.²

For a given partition S , the agent chooses a consumption plan (a vector c in $\mathbb{R}_+^N$) that is S -measurable, that is, if states i and j are in the same cell of S then $c_i = c_j$. Let $\mathcal{C}_S$ be those plans;

²We assume that k is a fixed constraint. An alternative way to represent imperfect categorization would be to postulate that there is a cost for using each partition. A similar distinction occurs in models of rational inattention, c.f., [Sims \(2003\)](#) and [Woodford \(2012\)](#). We discuss this model in more detail in Section 5.2.

let $p \in \mathbb{R}_+^N$ be the price, let $w \in \mathbb{R}_+$ be the agent's wealth and let $B(p, w) = \{c \in \mathbb{R}_+^N : p \cdot c \leq w\}$ be her budget. Then, the consumer solves the following maximization problem:

$$\max_{S \in \mathcal{P}(k)} \max_{c \in \mathcal{C}_S \cap B(p, w)} \sum_{i=1}^n u(c_i) \pi_i, \quad (1)$$

where π_i is the prior probability of state i . Thus, consumers in our model make two choices: first they choose which states to pay attention to (choice of a partition $S \in \mathcal{P}(k)$); second, they choose an optimal consumption consistent with their attention strategy and their budget (the choice $c \in \mathcal{C}_S \cap B(p, w)$). The inner maximization is a standard consumer problem with a coarser state space. The outer maximization problem represents optimal attention allocation: the agent optimally categorizes the states, paying attention to the distinctions between some states but ignoring others.

An alternative and simpler way to state the consumer's optimization problem is to restate the complexity constraint as a constraint on feasible consumption plans. Let $\mathcal{C} = \mathbb{R}_+^N$ be all consumption plans. Any $c \in \mathcal{C}$ that is measurable with respect to a k -cell partition has at most k distinct consumption levels and, conversely, for any $c \in \mathcal{C}$ with k or fewer distinct consumption levels there is some partition $S \in \mathcal{P}(k)$ that renders the consumption plan c feasible. This motivates the following definition:

Definition 1. The consumption plan $c \in \mathcal{C}$ is *coarse* if the collection $\{c_i\}_{i \in N}$ has at most k elements.

We write $\mathcal{C}_k$ for the set of all coarse consumption plans. The complexity constraint can now be restated as the assumption that agents are restricted to *coarse consumption plans*; that is, plans with at most k distinct consumption levels. A consumer who is subject to both the budget constraint and the coarseness constraint must choose her consumption from the set

$$B_k(p, w) = \left\{ c \in \mathcal{C}_k : p \cdot c \leq w \right\}.$$

Let $U(c) := \sum_{i=1}^n u(c_i) \pi_i$. The consumer problem (1) can now be restated in the more compact form below:

$$\max_{c \in B_k(p, w)} U(c). \quad (2)$$

We assume that u is a strictly concave CRRA utility index; that is, for $\rho > 0$,

$$u(c_i) = \begin{cases} c_i^{1-\rho}/(1-\rho) & \text{if } \rho \neq 1 \\ \ln c_i & \text{if } \rho = 1. \end{cases}$$

Let $D_k(p, w)$ denote the set of optimal plans, that is, solutions to (2) under the assumption that u is a CRRA utility index.

The set of all partitions with k cells has a high cardinality; however, as we show in Theorem 1 below, we may restrict attention to partitions that correspond to price ranges.

Definition 2. A consumption plan c is *monotone in price* if $\frac{p_i}{\pi_i} > \frac{p_j}{\pi_j}$ implies $c_i \leq c_j$. The plan c is *measurable in price* if $\frac{p_i}{\pi_i} = \frac{p_j}{\pi_j}$ implies $c_i = c_j$.

The following theorem shows that the agent always chooses a consumption plan that is monotone and measurable with respect to the price, i.e., she consumes more in states that have lower (normalized) prices and never distinguishes states that have the same price.

Theorem 1. Any $c \in D_k(p, w)$ is monotone and measurable in price.

To provide intuition for Theorem 1 consider a consumer with log utility and wealth 1. For a given partition S of the states into k -cells, let $p_S = (p_S(1), \dots, p_S(k))$ and $\pi_S = (\pi_S(1), \dots, \pi_S(k))$ be the prices and probabilities of the k -cells. That is, $p_S(l) = \sum_{j \in S_l} p_j$ and $\pi_S(l) = \sum_{j \in S_l} \pi_j$. The standard expression for the indirect utility of a log consumer implies that the utility induced by the partition S is $d_1(\pi_S \parallel p_S)$, the relative entropy of π_S with respect to p_S :

$$d_1(\pi_S \parallel p_S) := \sum_{j=1}^k \pi_S(j) \ln \left(\frac{\pi_S(j)}{p_S(j)} \right).$$

Thus, we can interpret the consumer's problem as choosing a partition that maximizes the relative entropy of the cell probabilities π_S with respect to the cell prices p_S . The relative entropy is a *convex* function of (π_S, p_S) . In the proof of Theorem 1 we use this convexity to show that all optimal partitions must be monotone and measurable.³ For example, to prove measurability, consider any partition that splits states with identical normalized prices.

³The induced utility over (π_S, p_S) is convex for all CRRA utility functions and for CARA utility.

This partition leads to prices and probabilities that are a convex combination of prices and probabilities of the partitions that do not split states. Convexity of the induced utility implies that such a convex combination cannot be optimal.

Theorem 1 simplifies the attention allocation problem. As an illustration of the optimal attention choice, consider the case of three equally likely states, log utility, and wealth 1. The states are ordered so that $p_1 > p_2 > p_3$. Then, the consumer either lumps together states 1 and 2, or states 2 and 3. Using the above expression, the induced utility of the first partition is $-\frac{2}{3} \ln(p_1 + p_2) - \frac{1}{3} \ln p_3 - \ln 3$ and the second partition is $-\frac{1}{3} \ln p_1 - \frac{2}{3} \ln(p_2 + p_3) - \ln 3$. If

$$\frac{p_1}{p_2} \leq \frac{p_2}{p_3} \quad (3)$$

the optimal plan combines states 1 and 2 as “bad times” with the same low consumption in each and identifies state 3 as “good times” with high consumption. If inequality (3) is reversed, the consumer chooses low consumption in state 1 and the same high consumption in states 2 and 3. If (3) holds with equality there are two optimal consumption plans corresponding to distinct attention strategies. A convex combination of those two plans would yield higher utility but is infeasible because it would require paying attention to all three states.

Theorem 1 relies on the convexity of the induced utility over cell prices and cell probabilities. This convexity holds for constant relative risk aversion but also for constant absolute risk aversion. However, it is not true for all concave utility indices. For example, suppose there are four equally likely states with $p_1 = 2, p_2 = p_3 = 1$ and $p_4 = 1/2$. The consumer has wealth 6, $k = 2$, and a piecewise linear utility index: for $0 < c_i < 1$ marginal utility is very large; for $1 \leq c_i \leq 2$ marginal utility is 1 and for $c_i > 2$ marginal utility is close to zero. It is straightforward to verify that no optimal plan combines states 2 and 3 as required by Theorem 1.⁴

⁴This example can be modified to show that the optimal plan may be non-monotone. Suppose the price of consumption in state 3 is $1 + \epsilon$ and all other prices stay unchanged. Suppose, further, the probability of state 2 is raised slightly at the expense of state 3. Then, (for ϵ sufficiently small) we can find a wealth w such that the consumer strictly prefers to combine states 1 and 3 over all other plans.

3 Coarse Competitive Equilibria

We analyze an endowment economy consisting of a continuum of agents who share a common utility function and a common prior π but have idiosyncratic endowments. Because utilities are homogenous (the utility index is constant relative risk aversion) the distribution of endowments has no effect on equilibrium prices. For this reason, we perform our analysis for a representative consumer who holds the per capita endowment. This representative consumer is an analytical device and not a reflection of the actual economy.

The vector $s = (s_1, \dots, s_n)$ represents the aggregate endowment. We write a for the smallest value of s_i , b for the largest value of s_i , and assume that $0 < a < b$. If the aggregate endowment has k or fewer distinct values, then the consumer's complexity constraint does not bind. In that case, the economy has a standard competitive equilibrium in which the representative agent consumes the aggregate endowment. Conversely, if $k = 1$, that is, if all consumers must choose the same consumption in every state, then the only feasible allocation has the representative agent consume a , the minimum endowment, in every state. Thus, to avoid these trivial outcomes, we assume that k is greater than 1 but smaller than the number of distinct values of s .

Even though we can assume a representative consumer, this does not mean that all agents' consumption plans are the same (or a multiple of the same consumption plan). If they were, then the aggregate consumption plan would take on at most k distinct values. Since the aggregate endowment has more than k distinct values this would mean that the endowment in some states of the world is wasted. As we show below, this is typically not optimal and, therefore, ex ante identical consumers must choose distinct consumption plans. To model this, we describe an allocation as a finite collection of distinct consumption plans and weights for each plan that represent the share of the population that chooses that plan. More precisely, let Δ denote the set of functions $\mu : \mathcal{C}_k \rightarrow [0, 1]$ such that the support of μ , denoted by $K(\mu) = \{c : \mu(c) > 0\}$, is finite and $\sum_{c \in \mathcal{C}_k} \mu(c) = 1$. An *allocation* is a $\mu \in \Delta$ where $\mu(c)$ represents the fraction of agents who choose plan $c \in \mathcal{C}_k$. The allocation μ is *feasible* if the average consumption is less

than or equal to the per capita endowment:

$$\sigma_i(\mu) := \sum_{c \in K(\mu)} c_i \cdot \mu(c) \leq s_i$$

for all $i \in N$. Let $\Phi \subset \Delta$ be the set of all feasible allocations.

Definition 3. A *coarse competitive equilibrium (CCE)* is a price p and an feasible allocation $\mu \in \Phi$ such that all plans in the support of μ solve the maximization problem (2) at prices p and wealth $s \cdot p$; that is, $\mu(c) > 0$ implies c is optimal for the representative agent, that is $c \in D_k(p, s \cdot p)$.

Two consumption plans c, c' *conform* if they correspond to the same partition of the state space, that is, $c_i = c_j$ if and only if $c'_i = c'_j$. The following four properties of equilibrium play a key role in our analysis:

Definition 4. An allocation μ is

- (i) *simple* if $c, c' \in K(\mu), c \neq c'$ implies that c and c' do not conform.
- (ii) *fair* if $c, c' \in K(\mu)$ implies $U(c) = U(c')$.
- (iii) *monotone* if for all $c \in K(\mu)$, $c_i \geq c_j$ whenever $s_i > s_j$.
- (iv) *measurable* if for all $c \in K(\mu)$, $c_i = c_j$ whenever $s_i = s_j$.

In a simple allocation, each categorization of the states has at most one consumption plan associated with it. Thus, if μ is simple, the cardinality of $K(\mu)$ is at most equal to the number of partitions in $\mathcal{P}(k)$. A fair allocation yields the same utility to every agent. A monotone allocation is one where consumption is weakly increasing in the aggregate endowment. A measurable allocation is an allocation that remains feasible if states with identical endowments are combined into a single state.

The mean utility, $W(\mu)$, of allocation μ is

$$W(\mu) = \sum_c U(c) \cdot \mu(c).$$

We say that $\mu \in \Phi$ solves the *planner's problem* if $W(\mu) \geq W(\mu')$ for all $\mu' \in \Phi$. The main result of this section is Theorem 2 below, which identifies properties of the solutions to the planner's problem and relates the planner's problem to CCE equilibria of the representative economy. In an economy without our complexity constraints, simplicity, fairness, monotonicity and measurability of solutions to the planner's problem would follow immediately from the strict concavity of the utility function. The argument for simplicity is unaffected by behavioral constraints, as for any two plans measurable with respect to the same partition, their average is also measurable and yields a higher utility for the planner. However, none of the remaining properties hold for a general strictly concave utility function given the coarseness constraint. Theorem 2 shows that they do hold with a strictly concave CRRA utility function.

Theorem 2. *(i) There is a solution to the planner's problem and every solution to the planner's problem is simple, fair, monotone, and measurable. (ii) An allocation solves the planner's problem if and only if it is a CCE allocation.*

Existence of a CCE and its Pareto-efficiency relies neither on CRRA preferences nor on the correspondence between solutions to the planner's problem and equilibria; as long a utility functions are continuous, existence of a CCE can be established using a fixed-point argument and as long as preferences are locally non-satiated, the first welfare theorem holds. (Of course, Pareto efficiency is defined relative to allocations in Φ .) Such a proof would not yield the monotonicity and measurability of equilibrium allocations. As we show in Section 5, without CRRA utility, it is possible to construct examples of CCE that do not satisfy these properties.

Next, we show that the representative consumer bears greater risk in a CCE than in the corresponding standard economy without the coarseness constraint. Notice that we can interpret the equilibrium allocation as a lottery over distinct consumption plans. The representative consumer is indifferent as to how this lottery resolves because she is indifferent between all plans in the support of the equilibrium allocation. Once a particular plan is chosen, a second round of randomization yields the state and the corresponding consumption. In a standard economy without the coarseness constraint, the representative consumer simply consumes her endowment in every state and therefore faces only the second round of randomization. To compare those two allocations in terms of their implied risk, it is useful to interpret the CCE

consumption lottery in reverse order where we first choose the state and only then choose the particular consumption plan in the support of μ . Note that in each state the expected value of the consumption lottery is weakly below s_i . As a result, the consumption lottery in the standard economy *second order stochastically dominates* the CCE consumption lottery, and therefore the representative consumer bears greater consumption risk in a CCE than in a SCE.⁵

The final result of this section shows that equilibrium prices are essentially unique and monotone. A price p is *monotone* if greater aggregate endowment implies a weakly lower price, that is, $\frac{p_i}{\pi_i} \leq \frac{p_j}{\pi_j}$ whenever $s_i > s_j$. In a *pure endowment economy* the realized endowment resolves all uncertainty; that is, $s_i \neq s_j$ for all states i and j . In that case, the equilibrium price is unique and monotone. If there is more than one state with a given endowment, the equilibrium price need not be unique; however, the *sum* of the prices for a given endowment is unique. For any $r \in \{s_i : i \in N\}$, let $p(r) = \sum_{\{i:s_i=r\}} p_i$ and $\pi(r) = \sum_{\{i:s_i=r\}} \pi_i$. Two prices $p, \hat{p}$ are *equivalent* if $p(r) = \hat{p}(r)$ for all r . We say that the price is *essentially unique* if all equilibrium prices are equivalent.

Theorem 3. *The CCE price of any economy is essentially unique and monotone.*

To illustrate CCE prices and consumptions, consider the following example. The utility function is logarithmic, there are four equally likely states and $k = 2$. Table 1 below describes the aggregate endowment s , the three coarse consumption plans (c^1, c^2, c^3) in the support of the equilibrium allocation, the CCE equilibrium price and, finally, the equilibrium price in a standard economy without the coarseness constraint.

endowment	c^1	c^2	c^3	CCE price	std. price
1	.85	1.04	1.18	.40	.35
4/3	1.7	1.04	1.18	.25	.26
5/3	1.7	1.96	1.18	.21	.21
2	1.7	1.96	2.56	.13	.18

Table 1: Equilibrium allocations and prices.

⁵By focusing on the representative consumer we ignore the wealth effects: if agents have different initial endowments, some of them may have a greater share of the total wealth in the CCE economy than in the standard economy and this wealth effect may be large enough to offset the effect of greater consumption volatility for a *given* wealth share.

Note that equilibrium consumption plans are characterized by a cutoff state; that is, each agent chooses a state $j \in \{1, 2, 3\}$ such that states $i \leq j$ are associated with low consumption (“bad times”) and states $i > j$ are associated with high consumption (“good times”). The last column of Table 1 contains the equilibrium price without the coarseness constraint in which the representative consumer simply consumes the aggregate endowment. Notice that the largest difference between the equilibrium price in a standard economy and the CCE price is in the extreme states with endowments 1 and 2. In the standard case, the price ratio between those states is 2, the ratio of the aggregate endowment in those two states. By contrast, the price ratio between those same states is 3 in a CCE. As we will show in the next section, this is no accident. In any CCE with many states, the prices in states with endowments near the upper or lower bounds exhibit the greatest departure from standard equilibrium prices.

4 Extreme Prices

The four state example of the previous section suggests that the effect of coarseness on equilibrium prices is most pronounced when the endowment realization is either unusually large or unusually small. Theorem 4, our main result, shows that this effect is true in general. In Theorem 4 we consider an economy that closely approximates an economy with a continuous endowment distribution. In other words, we consider a sequence of economies where we hold fixed the utility index u , the coarseness constraint k and vary the aggregate endowment. Our main result characterizes the limit of the corresponding equilibrium prices near the upper and lower boundary of the endowment range.

Let E^n be a pure endowment economy with $n \geq k + 1$ states and order states so that $s_i < s_j$ if $i < j$. A sequence of economies $\{E^n\} = \{(u, k, \pi^n, s^n)\}$ is *convergent* if s^n converges in distribution to a probability distribution with a continuous and strictly positive density on $[a, b]$. Let p^n be the equilibrium price of E^n . Some of properties of the limit price are more conveniently stated in terms of the pricing kernel κ^n , that is, the equilibrium price normalized by the probability of the state. The *pricing kernel* is

$$\kappa_i^n = \frac{p_i^n}{\pi_i^n}$$

We call the sequence $\{(p^n, \kappa^n)\}$ of a convergent sequence of economies a *CCE price sequence*.

For any real-valued function X on $\{1, \dots, n\}$ and $A \subset \mathbb{R}$, let $\Pr(X \in A)$ denote the probability that X takes a value in A ; that is, $\Pr(X \in A) := \sum_{\{i: X_i \in A\}} \pi_i$. In the statements below, we use the fact that any sequence $\{x^n\} \subset [0, 1]$ has a convergent subsequence and write $\lim x^n$ to denote any such limit.

Theorem 4. *For any CCE price sequence $\{(p^n, \kappa^n)\}$*

- (i) *$\lim p_1^n > 0$ and, if $\rho \geq 1$, then $\lim \Pr(\kappa^n > K) > 0$ for all K ;*
- (ii) *$\lim \Pr(\kappa^n < \epsilon) > 0$ for all $\epsilon > 0$ and, if $\rho > 1$, then $\lim \Pr(p^n = 0) > 0$.*

For the state with the lowest endowment, Theorem 4(i) establishes that the limit price is greater than zero even though the limit probability of that state is zero. Thus, consumption in the lowest endowment state is extremely expensive. Clearly, this implies that the pricing kernel in state 1 diverges but since the probability of state 1 converges to zero, this leaves open the question of whether there is a positive limit probability of an arbitrarily high pricing kernel. The second part Theorem 4(ii) shows that this is the case if the agent is sufficiently risk averse, with a parameter of relative risk aversion greater or equal to 1.

Part (ii) of Theorem 4 establishes that there is a positive limit probability that the pricing kernel is arbitrarily close to zero. By Theorem 3 this occurs when the endowment realization is near its upper bound b . Finally, if relative risk aversion is above 1, a stronger result is true: the limit price is zero in a nontrivial interval of states near the highest endowment realization.

To prove Theorem 4, we first establish the following dominance lemma (Lemma 12): let U_*^n be the maximal utility of a representative consumer in E^n at the CCE price when restricted to k -coarse plans. Let Y^n be the maximal utility of the same agent at the same price but under the restriction to $k - 1$ -coarse consumptions. Clearly, $Y^n \leq U_*^n$. The dominance lemma shows that Y^n is uniformly bounded away from U_*^n for all n ; hence $k - 1$ -coarse plans do uniformly worse than equilibrium plans.

To see the argument for the first part of Theorem 4, assume that the equilibrium price in state 1 converges to zero. In equilibrium, some agents must choose a lower consumption in state 1 than in all other states because aggregate consumption is lower in state 1 than in all

other states and because all equilibrium plans are monotone. An alternative plan for these consumers would be to increase consumption in state 1 and make it equal to consumption in state 2 while reducing consumption in the remaining states a bit so as to satisfy the budget constraint. If the price in state 1 converges to zero, then this plan yields essentially the same utility as the original plan, as the compensating reduction in consumption in higher states vanishes. But since the new plan is $k - 1$ coarse we have established a contradiction to the dominance lemma. Hence, the price in state 1 must stay bounded away from zero.

A similar application of the dominance lemma shows that consumption in the highest endowment states must be very cheap so that consumers find it worthwhile to single them out: so cheap that the probability-weighted utility in those states stays bounded away from zero. As a consequence, the pricing kernel must be close to zero. For the final part of Theorem 4 note that utility is bounded above if $\rho > 1$ and, therefore, consumers are unwilling to single out very unlikely low-price events no matter how low the price. In that case, part of the aggregate endowment near b is not consumed and prices are zero.

We can relate the quantity $p_1^* := \lim p_1^n$ to the value of relaxing the coarseness constraint. Suppose, at a cost τ , the agent can relax the coarseness constraint from k to $k + 1$. Thus, after this trade the agent has smaller wealth (by τ) but can choose a plan in $\mathcal{C}_{k+1}$. Suppose that $\tau < a \cdot p_1^*$ (where $a > 0$ is the endowment in state 1). In equilibrium, some agents must consume at least a in the lowest state. For such an agent, consider a new plan that isolates state 1 and consumes $\epsilon > 0$ in that state. In the limit, as $\epsilon \rightarrow 0$, this plan relaxes the budget constraint by $p_1^* a > \tau$. Since the probability of state 1 goes to zero, the utility of this plan is only slightly lower than the original utility and this difference vanishes as n goes to infinity. Thus, ap_1^* is bounded above by the shadow price of the coarseness constraint. It follows that p_1^* converges to zero as k goes to infinity: the equivalence of equilibrium allocations and solutions to the planner's problem established in Theorem 2 ensures that the value of an additional partition element must go to zero eventually. Since ap_1^* is smaller than the shadow price of additional partition elements, p_1^* must converge to zero as k goes to infinity.

In the special case where $k = 2$ and consumers have a log-utility, the limit price can be calculated as the solution of a simple differential equation. For $k = 2$, monotonicity means that for each agent, there is a cutoff state j and two consumption levels $x \leq y$ such that the agent

consumes x if the state is lower than j and consumes y otherwise. Since the utility function is unbounded, the price must be strictly greater than zero in every state, which implies that the feasibility constraint holds with equality. This, in turn, implies that for every j , there is a consumption plan in the support of the equilibrium allocation with cutoff j ; otherwise aggregate consumption would be the same in two consecutive states and since aggregate endowment is strictly increasing, feasibility would not be satisfied with equality.

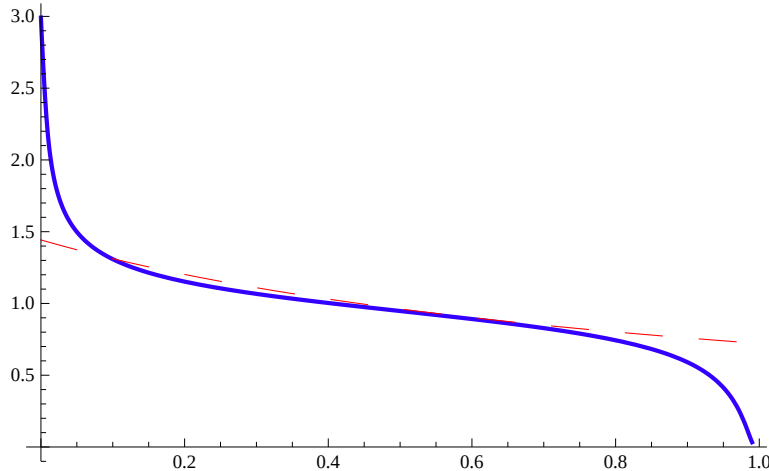


Figure 1: Limit price in CCE and in standard economy

Figure 1 depicts this limit price for the case where the limit endowment is uniformly distributed on $[a, b]$. Because of the uniform distribution the equilibrium price is proportional to the pricing kernel. The solid line represents the CCE price and the dashed line represents the equilibrium price in a standard economy without the coarseness constraint. From Theorem 4 it follows that the price diverges as the endowment converges to the lower bound a and to zero as the endowment converges to the upper bound b .

5 Robustness and Extensions

In this section, we examine the robustness of our results to various modeling assumptions. Specifically, we examine how our results would change if we allowed agents to be differentiated by their risk posture and their complexity constraint; if instead of coarse consumption we assumed a coarseness constraint on net trades; and, finally, to what extent our conclusions

depend on the assumption of constant relative risk aversion. We also briefly discuss an extension of our model to infinite horizon problems and implications of our model for asset prices.

5.1 Differentiated Households

We have assumed that all agents share a common CRRA utility index and a common complexity constraint k . Instead, consider a model with a finite set of types, each with a type-specific CRRA utility index and a type-specific complexity constraint k . Monotonicity and measurability in price (Theorem 1) continues to hold for each agent. Moreover, existence of equilibrium can be established using a standard existence argument for continuum economies.⁶ Because aggregate demand is monotone and measurable, it follows that equilibrium prices continue to be monotone in the aggregate endowment. That is,

$$s_i > s_j \text{ implies } \frac{p_i}{\pi_i} \leq \frac{p_j}{\pi_j}$$

with a strict inequality if $p_j > 0$. Monotonicity of prices, in turn, implies that in a pure exchange economy all individual consumption plans are monotone and measurable in the aggregate endowment. Of course, the economy no longer has a representative consumer and the equilibrium allocation is no longer simple. However, simplicity and the existence of a representative consumer play no role in the proof of our main theorem, Theorem 4. Thus, a modified version of Theorem 4 would continue to hold: if *all types* satisfy the corresponding condition on the parameter of risk aversion then the conclusion of Theorem 4 remains unchanged.

5.2 Costs Instead of Constraints

We have assumed that agents cannot adjust their coarseness constraint. Consider, instead, the following extension of our model. Agents can choose k but incur a utility cost $c(k)$. Assume that $c(2) = 0$ so that agents have at least 2 elements in their partition. Further assume $c(k+1) - c(k) > 0$ for some $k \geq 2$ and that the marginal cost of relaxing the constraint is

⁶The only non-standard feature of our economy is the non-convexity of the consumption set. However, our non-convexity is benign in the sense that the demand correspondence remains upper-hemi-continuous and, therefore, a standard existence argument for continuum economies (Aumann, 1966) applies.

increasing, that is, $c(k+1) - c(k)$ is non-decreasing in k . For each k , the consumer solves the decision problem analyzed in section 2. In addition, the consumer chooses k to maximize her overall utility taking into account the utility cost of k .

For any fixed value of k individual demand continues to be monotone and measurable. Thus, Theorem 1 would continue to hold under this extension. Moreover, the optimal k would be smaller than n , the number of states when n is large and would stay bounded as we let n go to infinity. Of course, it may no longer be true that all consumers choose the same k in equilibrium. But, as we argued above, this would have no effect on Theorem 4, our main result. Thus, while we have not formulated the model with cost, we conjecture that our main result would extend unchanged to the case where agents choose k optimally and the utility cost of k satisfies the above conditions.

5.3 Alternative Coarseness Constraint

We require coarse consumption plans but we allow agents to take full advantage of complete markets to finance their plans.⁷ This is similar to other models of inattention, such as [Reis \(2006\)](#). This assumption is reflected in the fact that consumers face a single budget constraint and can transfer wealth between states without additional restrictions. Our justification for this assumption is pragmatic: the assumption of unrestricted financial transactions is what makes our model tractable.

An alternative (and more difficult) model would focus on the financial side of consumer behavior and require that consumers choose coarse *trading strategies*. One way to implement such a constraint would be to require that *net trades*, that is, the difference between consumption and endowments, be coarse. For example, assuming agent's net trades are 2-coarse amounts to assuming that states are partitioned into "borrowing states" and "lending states" and the agent borrows some fixed amount x whenever it finds herself in a borrowing state and lends a fixed amount y if she finds herself in a lending state.

⁷As an alternative to consumption rigidities, some authors introduce market incompleteness to achieve similar goals. [Constantinides and Duffie \(1996\)](#) introduce a model where consumers cannot insure their idiosyncratic income shocks and [Krebs \(2004\)](#) examines the observable implications of a generalization of their model. Even though our model retains the assumption of complete markets, it nevertheless delivers an equilibrium in which marginal rates of substitution are not equal across consumers and, therefore, shares a key implication with the incomplete markets literature.

In that case, the predictions on equilibrium prices are sensitive to the distribution of initial endowments. A plausible hypothesis is that each consumer's initial endowment is itself coarse, that is $\omega_i \in \mathcal{C}_k$ for all i .⁸ Now assume that, in addition, agents are restricted to k' -coarse net trades. Then, a consumer's feasible consumption plans $\hat{\mathcal{C}}(\omega)$ are a subset of $\mathcal{C}_{\hat{k}}$ where $\hat{k} = k \cdot k'$. Thus, the difference from our model is that consumers cannot achieve all consumptions in $\mathcal{C}_{\hat{k}}$ but only an endowment-dependent subset. By carefully choosing individual endowments we can construct an economy with k' -coarse net trades that has the same equilibrium allocation (and prices) as the economy with $\hat{k}$ -coarse consumption. This illustrates that restrictions on net trades together with the assumption of coarse endowments can yield extreme prices as characterized by Theorem 4. On the other hand, if we choose individual endowments to coincide with the unconstrained equilibrium allocation, that is, if individual endowments are proportional to the aggregate endowment, then a net-trades model would replicate the standard equilibrium. In this case, the distribution of initial endowments subverts any effect the coarse net trades might have.

As is apparent from the discussion above, there is a modeling tradeoff between the two types of constraints. Which constraint is appropriate depends on the particular application; specifically, it depends on what is the agent's *active decision*, i.e., the decision that is the focus of the agent, and what are the *residual decisions*, i.e., decisions that the agent is not focusing on, but are automatically implied by the active decision and the budget constraint.

The net-trades model is attractive because we can interpret it as a description of the agent's financial transactions. Its weakness is that the distribution of initial endowments plays a key role in the analysis. If endowments are coarse then there are settings that replicate our extreme-price results in an economy with coarse net trades. How generally this holds true is an open question left for future research.

⁸Of course, the *aggregate endowment* is not k -coarse. For example, suppose that each agent's endowment is either h or l where h represents her endowment if she is employed and l represents unemployed benefits. The aggregate endowment is $\alpha(s)h + (1 - \alpha(s))l$ where $\alpha(s)$ is the fraction of employed agents in state s which can be strictly increasing in s .

5.4 Non CRRA

The following example illustrates how equilibrium consumption plans may be non-monotone when the utility index is not CRRA. Specifically, suppose the utility index is given by

$$u(z) = \begin{cases} 2z & \text{if } z \leq 1 \\ 1 + z & \text{if } z \in [1, 2] \\ 2 + z/2 & \text{if } z > 2. \end{cases}$$

Table 2 summarizes the unique CCE equilibrium for the aggregate endowment $s = (1, 4/3, 5/3, 2)$. The table indicates that, in equilibrium, agents choose one of two consumption plans, c^1 or c^2 . The fractions in parenthesis indicate the fraction of agents choosing the respective plans.

endowment	c^1 (2/3)	c^2 (1/3)	CCE price	std. price
1	1	1	.25	.25
4/3	1	2	.25	.25
5/3	2	1	.25	.25
2	2	2	.25	.25

Table 2: Equilibrium allocations and prices

Notice that the second consumption plan c^2 is not monotone establishing that with general risk averse utility functions monotonicity may fail.⁹ However, CRRA is not the only class of utility functions that yields monotone consumption plan. If utilities are constant *absolute* risk averse, monotonicity would continue to hold. Characterizing the set of utility functions for which CCE consumption plans are monotone remains an open question.

5.5 Dynamic Decision Problems

We have conducted our analysis for a two-period model. In this section, we briefly discuss how coarse consumption can be extended to infinite-horizon economies. The formal analysis is in the supplementary online appendix. There, we analyze a setting in which endowments evolve

⁹The utility function in the example is not strictly concave. However, it is straightforward to show that a strictly concave approximation of the utility function in this example would also lead to non-monotone equilibrium consumption plans.

according to a finite state Markov process and consumers maximize discounted expected utility with a CRRA utility index. At an initial stage, each agent partitions the set of possible histories into finitely many categories. For any two histories in a given category the agent chooses the same consumption.

We show in the supplementary appendix that the consumer’s optimal attention strategy will partition the range of prices into time-invariant intervals. This characterization allows us to interpret coarse consumption plans as the following two-step optimization problem. In the first step, the consumer identifies k *price intervals*¹⁰ with the interpretation that she will not pay attention to price variations within each interval. The consumption choice then solves a standard dynamic optimization problem given the chosen price ranges, using the *average price* for each interval.

Optimal strategies are particularly simple if prices themselves are stationary, that is, if the pricing kernel is the same every time a given state occurs. In that case, each optimal attention strategy is monotone not only in prices, but also in states and thus uses the same partition of states every period. As we show in the online appendix, such stationary equilibria exist and extend coarse competitive equilibria to an infinite horizon while retaining the tractability of the two-period model.

5.6 Safe Haven Premium and Extreme Asset Prices

Consider an asset that is “almost” risk free, that is, it pays off one unit of consumption with probability $1 - \epsilon$ and pays of nothing with probability ϵ . The set of states where the asset pays of zero are those states where consumption is most expensive. Clearly, the risk-free bond that pays off one unit of consumption in all states will trade at a premium over the nearly risk free bond. However, in a standard competitive equilibrium this premium converges to zero as ϵ converges to zero. By contrast, this premium stays positive, even in the limit as $\epsilon \rightarrow 0$, in the economy with coarse consumers. This observation is a straightforward corollary of Theorem 4: in the limit economy with a continuous state space, the price of one unit of consumption in ϵ –most

¹⁰The relevant price is the price of consumption after any history appropriately normalized. The appropriate normalization divides the (ex-ante) price of consumption by the probability of the particular history and the discount factor. See our online appendix for details.

expensive states stays bounded away from zero for all ϵ . Thus, an asset that is *guaranteed* to pay off is different from an asset that pays off with near certainty.

Thus, a truly risk-free bond captures a *safe-haven premium* over a nearly risk-free bond with a very small default risk. From the perspective of a coarse consumer, this premium captures the benefit of “not having to pay attention to” the possibility of default.

In an infinite horizon model with coarse consumers (see the online appendix) we can translate Theorem 4 into a statement about asset prices: in states where the aggregate endowment is extreme (that is, near the upper or near the lower bound) asset prices, that is, claims on future endowments will also be extreme. As the endowment converges to the upper bound, asset prices (expressed in terms of current consumption) diverge and at the lower bound they converge to zero. Thus, unusually high realizations of the aggregate endowment are associated with extremely high asset prices while unusually low realizations of the aggregate endowment are associated with extremely low asset prices.

6 Conclusion

In this paper, we analyzed the implication of coarse consumption plans on equilibrium prices in a standard endowment economy. We assumed that agents’ consumption plans must be k -coarse, that is, may take on at most k distinct values. We showed that the coarseness constraint leads to extreme and volatile prices when the endowment realization is near its upper or lower bound.

Many of the empirical puzzles in macroeconomics and finance arise from researchers’ inability to reconcile the levels of risk aversion implied by equilibrium models with levels that are observed in other contexts or are reasonable a priori (Mehra and Prescott, 1985; French and Poterba, 1991). Our work draws a distinction between the basic preferences that describe behavior in absence of any additional constraints and the revealed preferences that describe behavior that has been filtered through cognitive constraints, such as those considered in this paper. We show that the latter preferences may exhibit more risk aversion than the former. While in some circumstances we may not care about this difference, in others, such as when considering policy interventions that will affect the filter, this distinction will be important.

There are a number of other papers that use rigidities in consumption to close the gap

between the level of risk aversion needed to rationalize data and plausible levels of risk aversion. Grossman and Laroque (1990) distinguish liquid and illiquid consumption and assume that agents incur transaction costs when they sell an illiquid good. Chetty and Szeidl (2010) focus on the extent to which consumption rigidities reduce stock market participation. Lynch (1996) and Gabaix and Laibson (2002) study a model in which only a fraction of agents can make adjustments at a given time. Unlike those two papers, we do not exogenously fix the fraction of agents that can respond to a particular increase in aggregate output, but rather we make adjustments optimal subject to an attention constraint.

The fact that agents choose their partitions endogenously enhances the impact of their behavioral limitation on equilibrium prices. This is because in settings where agents' attention allocation is responsive to incentives, the market mechanism acquires a new function: it allocates agents' scarce attention. For markets to clear, the equilibrium prices have to accentuate the relevant events to attract the agents' attention. Since it is particularly costly to pay attention to tail events, the price variation in those events has to be large enough to make them salient.

A Appendix: Proofs

For any given partition S the agent is facing k composite states of the world with probabilities $\pi_S(1), \dots, \pi_S(k)$ and prices $p_S(1), \dots, p_S(k)$. The agent chooses a consumption function $c_S : \{1, \dots, k\} \rightarrow \mathbb{R}_+$ to maximize $\sum_{j=1}^k \pi_S(j) u(c_S(j))$ subject to $\sum_{i=1}^k p_S(j) c_S(i) \leq w$. Let $\tilde{V}(S)$ be the maximal value.

Let $\sigma = 1/\rho$ be the elasticity of substitution between states and define $\psi_1(t) := t \log(t)$ and $\psi_\sigma(t) := t^\sigma$ for $\sigma \neq 1$. A routine calculation for CES utility maximization shows that $\tilde{V}(S)$ is a monotone transformation of the function V such that $V(S) = \ln w + d_1(\pi_S \parallel p_S)$ if $\sigma = 1$ and $V(S) = \frac{w}{\sigma-1} (d_\sigma(\pi_S \parallel p_S))$ if $\sigma \neq 1$ where

$$d_\sigma(\pi_S \parallel p_S) := \sum_{j=1}^k p_S(j) \psi_\sigma \left(\frac{\pi_S(j)}{p_S(j)} \right).$$

For $\sigma \geq 1$ we have $p_S(j) > 0$ for all j and, therefore, $V(S)$ is well defined. For $\sigma < 1$, it is possible that $p_S(j) = 0$ while $\pi_S(j) > 0$. In that case, define $p_S(j) \psi_\sigma \left(\frac{\pi_S(j)}{p_S(j)} \right) = 0 = \lim_{x \rightarrow 0} x \psi_\sigma \left(\frac{\pi_S(j)}{x} \right)$. This extension preserves continuity and captures the supremum of utilities attainable with any

feasible consumption plan. The following lemma notes a useful property of the function ψ_σ .

Lemma 1. *Suppose that $a_1, a_2, b_1, b_2 \geq 0$.*

1. *If $\sigma \geq 1$ and $b_1, b_2 > 0$, then $b_1\psi_\sigma\left(\frac{a_1}{b_1}\right) + b_2\psi_\sigma\left(\frac{a_2}{b_2}\right) \geq (b_1 + b_2)\psi_\sigma\left(\frac{a_1+a_2}{b_1+b_2}\right)$ with equality only if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.*
2. *If $\sigma < 1$ and $a_1 \cdot b_1, a_2 \cdot b_2 > 0$, then $b_1\psi_\sigma\left(\frac{a_1}{b_1}\right) + b_2\psi_\sigma\left(\frac{a_2}{b_2}\right) \leq (b_1 + b_2)\psi_\sigma\left(\frac{a_1+a_2}{b_1+b_2}\right)$ with equality only if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.*

Proof. For case 1, divide both sides by $(b_1 + b_2)$ and apply Jensen's inequality. For case 2 with positive b_1, b_2 do the same. Suppose $b_1 > 0, b_2 = 0$. Then we have $b_1\psi_\sigma\left(\frac{a_1}{b_1}\right) < b_1\psi_\sigma\left(\frac{a_1+a_2}{b_1}\right)$, which follows from monotonicity of ψ_σ . In this case equality never obtains, as $\frac{a_1}{b_1} < \frac{a_2}{b_2} = \infty$. The case $b_1 = 0, b_2 > 0$ is symmetric. Finally, the case of $b_1 = b_2 = 0$ leads to an equality while at the same time $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. $\square$

Thus, the preference of the agent over the set of all partitions is represented by a *divergence* of π with respect to p . In the case $\sigma = 1$ it's the Kullback–Leibler divergence (the relative entropy); in the case when $\sigma \geq 1$ it's the Hellinger divergence. An important property of V is that it is monotone: if the partition S' is a refinement of S , then $V(S') \geq V(S)$. This is a simple consequence of Lemma 1. Another consequence of it is that if the price kernel is strictly decreasing and $n > k$, then the cognitive constraint is strictly binding. Our dominance lemma is a uniform version of this result. Another property of V is that it is quasi convex, which helps us prove Theorem 1.¹¹

A.1 Proof of Theorem 1

We consider a relaxed problem for the consumer, in which he can split each state arbitrarily finely to create generalized partitions that contain fractions of states. Let m be the number of distinct normalized prices and let $\tilde{\pi}_i$ be the probability of price $i \in \{1, \dots, m\}$. Order the prices so that $p_1/\tilde{\pi}_1 > p_2/\tilde{\pi}_2 > \dots > p_m/\tilde{\pi}_m$. The set of *generalized partitions* is defined by

$$\mathcal{X} = \left\{ \xi \in [0, 1]^{mk} : \sum_{j=1}^k \xi_{ij} = 1 \text{ for all } i = 1, \dots, m \right\}.$$

¹¹We note that if u belongs to the CARA class, then the preference of the agent over the set of all partitions is also represented by the Kullback–Leibler divergence; thus, our results generalize to this case (with the lower bound on consumption removed to avoid corner solutions).

The vector $\xi_j = (\xi_{1j}, \dots, \xi_{mj})$ represents the “fractions” of state i that are part of cell j of the generalized partition ξ . In the actual problem, the agent chooses over the finite set of usual partitions of N . Its strict subset, partitions measurable with respect to the price, consists of vectors ξ that are integer valued, i.e., $\mathcal{P} = \{\xi \in \mathcal{X} : \xi_{ij} \in \{0, 1\}\}$.

Note that $\mathcal{P}$ is the set of extreme points of $\mathcal{X}$. The fact that any $\xi \in \mathcal{P}$ is an extreme point is straightforward. To see that there are no other extreme points, let $\xi \in \mathcal{X} \setminus \mathcal{P}$. Then there exist i, j such that $\xi_{ij} = \alpha \in (0, 1)$, which implies that there exists another s with $\xi_{is} = \beta \in (0, 1)$. Let ξ' be equal to ξ on all entries except for i, j and i, s ; $\xi'_{ij} = \alpha + \beta$; $\xi'_{is} = 0$. Let ξ'' be equal to ξ on all entries except for i, j and i, s ; $\xi''_{ij} = 0$; $\xi''_{is} = \alpha + \beta$. Note that $\xi', \xi'' \in \mathcal{X}$ and $\xi = \frac{\alpha}{\alpha+\beta}\xi' + \frac{\beta}{\alpha+\beta}\xi''$.

For any $\xi \in \mathcal{X}$ define $p_\xi(j) := \sum_{i=1}^m \xi_{ij} p_i$, $\pi_\xi(j) := \sum_{i=1}^m \xi_{ij} \tilde{\pi}_i$, and $U^*(\xi) := d_\sigma(\pi_\xi \parallel p_\xi)$. Since $\{1, \dots, k\}$ are just labels, for any optimal $\hat{\xi}$ there exists an equivalent permuted ξ for which the price kernel $p_{\xi(j)}/\pi_{\xi(j)}$ is decreasing in j . In fact, it must be strictly decreasing. To see that, suppose that ξ is such that $p_\xi(j)/\pi_\xi(j) = p_\xi(s)/\pi_\xi(s)$. Choose ξ' that combines cells j and s , i.e., ξ' is identical to ξ except that $\xi'_{ij} = \xi_{ij} + \xi_{is}$ and $\xi'_{is} = 0$ for all i . Clearly $V(\xi) = V(\xi')$, but $\pi_{\xi'}(s) = p_{\xi'}(s) = 0$, which we now show is never optimal. To see that, let j' be a cell that puts positive weight on at least two states i, l (it exists since $k < m$). Choose l to be the minimal state with $\xi'_{lj'} > 0$. Now choose ξ'' identical to ξ' except that $\xi''_{ij'} = \xi'_{ij'}$, $\xi''_{ij} = 0$, $\xi''_{lj'} = 0$, and $\xi''_{lj} = \xi'_{lj'}$. Note that $\frac{\pi_{\xi''}(j)}{p_{\xi''}(j)} = \frac{\tilde{\pi}_l}{p_l} < \frac{\pi_{\xi''}(j')}{p_{\xi''}(j')}$ because of the choice of l above. Then Lemma 1 implies that the quantity $U^*(\xi'') - U^*(\xi') = p_{\xi''}(j)\psi\left(\frac{\pi_{\xi''}(j)}{p_{\xi''}(j)}\right) + p_{\xi''}(j')\psi\left(\frac{\pi_{\xi''}(j')}{p_{\xi''}(j')}\right) - (p_{\xi''}(j) + p_{\xi''}(j'))\psi\left(\frac{\pi_{\xi''}(j) + \pi_{\xi''}(j')}{p_{\xi''}(j) + p_{\xi''}(j')}\right)$ is positive for $\sigma \geq 1$ and negative for $\sigma < 1$.

Let the set of *monotone* generalized partitions be defined as

$$\mathcal{M} := \{\xi \in \mathcal{X} : \xi_{ij} > 0 \text{ implies } \xi_{ls} = 0 \text{ for } s > j, l < i \text{ and for } s < j, l > i\}.$$

We now show that every optimal $\xi \in \mathcal{M}$. To see that, suppose that there are $j > s, i > l$ such that $\xi_{is} > 0, \xi_{lj} > 0$. Let $\delta > 0$ and choose $\hat{\xi}$, identical to ξ except that $\hat{\xi}_{ls} = \xi_{ls} + \delta \frac{\tilde{\pi}_i}{\tilde{\pi}_l}$, $\hat{\xi}_{ij} = \xi_{ij} + \delta$, $\hat{\xi}_{is} = \xi_{is} - \delta$, $\hat{\xi}_{lj} = \xi_{lj} - \delta \frac{\tilde{\pi}_i}{\tilde{\pi}_l}$. Choose δ small enough so that $\hat{\xi} \in \mathcal{X}$. Clearly, $\pi_{\hat{\xi}}(j) = \pi_\xi(j)$ for all j ; $p_{\hat{\xi}}(t) = p_\xi(t)$ for all $t \neq j, s$, $p_{\hat{\xi}}(j) + p_{\hat{\xi}}(s) = p_\xi(j) + p_\xi(s)$ and $p_{\hat{\xi}}(j) < p_\xi(j)$. Let $c_\xi(t)$ be the optimal consumption in cell $t \in \{1, \dots, k\}$ if the partition is ξ . Note that with CRRA utility, $c_\xi(t)$ is strictly increasing in t and, therefore, $\sum_{t=1}^k c_\xi(t) p_{\hat{\xi}}(t) < \sum_{t=1}^k c_\xi(t) p_\xi(t)$. Thus, with partition $\hat{\xi}$ the plan c_ξ is in the interior of the budget and, therefore, ξ cannot be optimal.

We now show that for $\sigma \geq 1$ the function $U^*(\xi)$ is convex and for $\sigma < 1$ the function $U^*(\xi)$ is concave. Moreover, the convexity/concavity is strict over the set $\mathcal{M}$. To see that, note that Lemma 1 implies that for $\sigma \geq 1$

$$\lambda p_\xi(j) \psi \left(\frac{\pi_\xi(j)}{p_\xi(j)} \right) + (1 - \lambda) p_\zeta(j) \psi \left(\frac{\pi_\zeta(j)}{p_\zeta(j)} \right) \geq (\lambda p_\xi(j) + (1 - \lambda) p_\zeta(j)) \psi \left(\frac{\lambda \pi_\xi(j) + (1 - \lambda) \pi_\zeta(j)}{\lambda p_\xi(j) + (1 - \lambda) p_\zeta(j)} \right),$$

and the reverse inequality for $\sigma < 1$. Summing over j delivers convexity (concavity) of U^* . Moreover, if $U^*(\lambda \xi + (1 - \lambda) \zeta) = \lambda U^*(\xi) + (1 - \lambda) U^*(\zeta)$, the Lemma 1 implies that $\frac{\pi_\xi(j)}{p_\xi(j)} = \frac{\pi_\zeta(j)}{p_\zeta(j)}$ for all j . Let l be the maximal i such that $\xi_{i1} > 0$ and l' be the maximal i such that $\zeta_{i1} > 0$. By monotonicity, this implies that $\xi_{i1} = 1$ for $i < l$ and $\zeta_{i1} = 1$ for $i < l'$. The equality $\frac{\pi_\xi(1)}{p_\xi(1)} = \frac{\pi_\zeta(1)}{p_\zeta(1)}$ and the strict monotonicity of the price kernel imply that $l = l'$ and moreover $\xi_{l1} = \zeta_{l1}$. The same argument applies to $j = 2$ and higher to deliver that $\xi = \zeta$.

Thus any optimal $\xi \in \mathcal{P} \cap \mathcal{M}$. This assertion proves that (i) the solution to the relaxed problem is a solution to the original problem and (ii) the solution to the original problem is monotone and measurable. $\square$

A.2 Proof of Theorem 2

If $K(\mu) \subset \{c^1, \dots, c^m\}$, we write $\mu = (\mathbf{a}, \mathbf{c})$ where $\mathbf{a} = (\alpha^1, \dots, \alpha^m)$, $\mathbf{c} = (c^1, \dots, c^m)$ and $\mu(c^l) = \alpha^l$ for all l . It will be understood that $\mathbf{a} = (\alpha^1, \dots, \alpha^m)$, $\hat{\mathbf{a}} = (\hat{\alpha}^1, \dots, \hat{\alpha}^m)$, and so forth. We write δ_c for the allocation μ where all agents consume $c \in \mathcal{C}_k$.

Lemma 2. *If μ is feasible and not simple, then there is a simple and feasible μ' such that $W(\mu') > W(\mu)$.*

Proof. Let $\mu = (\mathbf{a}, \mathbf{c})$. If μ is not simple, there is $c, c' \in K(\mu)$ such that c and c' conform. Let $c^* = \gamma \cdot c + (1 - \gamma) c'$ where $\gamma = \frac{\mu(c)}{\mu(c) + \mu(c')}$ and let μ^* be the allocation derived from μ by replacing c, c' with a $(\mu(c) + \mu(c'))$ probability of c^* . Since, c and c' are coarse, so is c^* and μ^* . Since u is strictly concave, $W(\mu^*) > W(\mu)$. Note that $|K(\mu^*)| < |K(\mu)|$. If μ^* is simple, we are done. Otherwise, repeat the above argument. Since $K(\mu)$ is finite, this process must terminate with a simple allocation. $\square$

Lemma 3. *If μ is feasible, simple but not fair, then there is a feasible, simple and fair μ' such that $W(\mu') > W(\mu)$ and $|K(\mu')| \leq |K(\mu)|$.*

Proof. Let $\mu = (\mathbf{a}, \mathbf{c})$, let x^l be the certainty equivalent of c^l and $\bar{x}^l$ be the corresponding constant consumption plan; that is, $u(x^l) = U(c^l)$ and $\bar{x}_i^l = x^l$ for all i, l . Also, let $x = \sum_{l=1}^m \alpha^l x^l$ and let $\bar{x}$ be the corresponding constant consumption plan. Let $\hat{\mu} = (\hat{\mathbf{a}}, \hat{\mathbf{c}})$ such that $\hat{\alpha}^l = \frac{\alpha^l x^l}{x}$ and $\hat{c}^l = \frac{x c^l}{x^l}$ for all l . Finally, let $\bar{\mu} = (\bar{\mathbf{a}}, \bar{\mathbf{c}})$ such that $\bar{\alpha}^l = \alpha^l$ and $\bar{c}^l = \bar{x}^l$ for all l . Since u is strictly concave and μ is not fair, $W(\delta_{\bar{x}}) > W(\bar{\mu})$. Since u is CRRA,

$$u^{-1}(U(\hat{c}^l)) = \frac{x}{x^l} u^{-1}(U(c^l)) = \frac{x}{x^l} x^l = x;$$

hence, $W(\hat{\mu}) = W(\delta_{\bar{x}})$. By definition, $W(\bar{\mu}) = W(\mu)$. Hence, $W(\hat{\mu}) > W(\mu)$. By construction $\hat{\mu}$ is fair. It is easy to verify that $\sum_l \hat{c}_i^l \hat{\alpha}^l = \sum_l c_i^l \alpha^l$ for all $i \in N$ and hence $\hat{\mu}$ is feasible. Clearly, $|K(\hat{\mu})| \leq |K(\mu)|$. $\square$

Lemma 4. *A solution to the planner's problem exists and every solution to the planner's problem is simple and fair.*

Proof. The allocation δ_c such that $c_i = \min_i s_i$ for all i is feasible. Thus, Φ is non-empty. Since δ_s second order stochastically dominates any feasible μ it follows that $W(\mu) < W(\delta_s)$ for every feasible $\mu \in \Phi$. Hence,

$$W_k = \sup_{\mu \in \Phi} W(\mu)$$

is well-defined. By Lemma 2 and Lemma 3, there exists a sequence of feasible, simple, and fair allocations $\mu^t = (\mathbf{a}^t, \mathbf{c}^t)$ such that $W(\mu^t) \geq W_k - 1/t$ and $\mathbf{a}^t \in \mathbb{R}_+^m$ for all t , where m is the cardinality of the set of all partitions of N with k or fewer elements.

By passing to a subsequence, $\mathbf{a}^t = (\alpha^{1t}, \dots, \alpha^{mt})$ converges to some $\mathbf{a} \in \Delta(\mathbb{R}_+^m)$. If c^{lt} is unbounded for some l , we must have $\alpha^l = 0$. Let $A \subset N$ be the set of l such that $\alpha^l \neq 0$. Then, $A \neq \emptyset$ and c^{lt} is bounded for all $l \in A$. Hence, there exists a subsequence of μ^t along which c^{lt} converges to some $c^l \in \mathcal{C}_k$ for every $l \in A$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ where $\mathbf{a} = (\alpha^l)_{l \in A}$ and $\mathbf{c} = (c^l)_{l \in A}$. Since $\lim W(\mu^t) = W_k$ and each μ^t is fair, $U(c^{lt}) = W(\mu^t)$. So, by the continuity of u , we have $U(c^l) = W_k$ for all $l \in A$ and therefore $W(\mu) = W_k$. Finally, $\sum_{l \in A} \alpha^{lt} c_i^{lt} \leq \sum_{l=1}^m \alpha^{lt} c_i^{lt} \leq s_i$ for all i, l, t and so $\sum_{l \in A} \alpha^l c_i^l \leq \sum_{l=1}^m \alpha^l c_i^l \leq s_i$ for all i, l . Hence μ is feasible and therefore μ solves the planner's problem. Then, Lemma 2 and Lemma 3 imply that μ must be simple and fair. $\square$

Lemma 5. *An allocation solves the planner's problem if and only if it is a CCE allocation.*

Proof. Let $Z = \mathbb{R}_{++}^N$. Then, for all $z \in Z$, let $\Phi^z(\mathcal{C}')$ be the set of all allocations with support contained in $\mathcal{C}'$ that are feasible for the economy $E = (u, k, \pi, z)$. Let $W_k(z) = \max_{\mu \in \Phi^z(\mathcal{C}_k)} W(\mu)$. Hence, W_k is the planner's value as a function of the endowment. Let $Z^s = \{z \in Z : W_k(z) > W_k(s)\}$.

Clearly, $W_k(z) > W_k(y)$ whenever $z_i > y_i$ for all $i \in N$ since we can take the optimal allocation for y and increase every consumption in every state by a small constant amount. Hence, Z^s is nonempty.

Suppose $|z_i - y_i| < \epsilon$ for all i . Let $y_i^+ = \max\{y_i, z_i\}$, $y_i^- = \min\{y_i, z_i\}$ for all i and let $\mu = (\mathbf{a}, \mathbf{c})$ be optimal for (u, π, y^+) . Then, $(\mathbf{a}, (1 - \frac{\epsilon}{a})\mathbf{c})$ is feasible for (u, π, y^-) . Since $W(\mathbf{a}, (1 - \frac{\epsilon}{a})\mathbf{c})$ is continuous in ϵ at $\epsilon = 0$ and $W_k(y)$ is nondecreasing in each coordinate, for $\epsilon' > 0$, there exists $\epsilon > 0$ such that $|W_k(y) - W_k(z)| \leq |W_k(y^+) - W_k(y^-)| < \epsilon'$ proving that W is continuous at y and hence Z^y is open.

We note that since W is a concave function of μ , W_k is a concave function of z and hence the set Z^s is convex. To see that, fix $z^1, z^2 \in Z^s$ and choose $\mu^i \in \Phi^{z^i}(\mathcal{C}_k)$ such that $W(\mu^i) = W_k(z^i)$ for $i = 1, 2$. By Lemma 1, such μ^i exist. Clearly, $\gamma\mu^1 + (1 - \gamma)\mu^2 \in \Phi^{\hat{z}}(\mathcal{C}_k)$ for $\hat{z} = \gamma z^1 + (1 - \gamma)z^2$ and hence $W_k(\gamma z^1 + (1 - \gamma)z^2) \geq W(\gamma\mu^1 + (1 - \gamma)\mu^2) = \gamma W(\mu^1) + (1 - \gamma)W(\mu^2) = \gamma W_k(z^1) + (1 - \gamma)W_k(z^2)$.

Since Z^s is nonempty, open, and convex, and $s \notin Z^s$, by the separating hyperplane theorem, there exists $p \in \mathbb{R}^n$ such that $p_i \neq 0$ for some i and $\sum_i p_i \cdot z_i > \sum_i p_i \cdot s_i$ for all $z \in Z^s$. Since W_k is nondecreasing in each coordinate, we must have $p_i \geq 0$ for every $i \in N$ and hence we can normalize p to ensure that $p \in \Delta(N)$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ be a solution to the planner's problem, where $\alpha^l > 0$ for all l . The argument establishing that each c^l must maximize U given budget $B(p)$ is standard and omitted.

Finally, to see that if (p, μ) is a CCE, then μ must be a solution to the planner's problem, note that since every agent has the same endowment, μ must be fair. But then, if μ did not solve the planner's problem, the solution to the planner's problem would Pareto-dominate it. However, it is straightforward to show that every CCE allocation is Pareto-efficient. $\square$

Lemma 6. *If (p, μ) is a CCE and $\sigma_i(\mu) < s_i$ then $p_i = 0$.*

Proof. If $c \in \mathcal{C}_k$ then $\gamma c \in \mathcal{C}_k$ for all $\gamma > 0$. Since U is strictly increasing it follows that $\sum_N p_i c_i = \sum_N p_i s_i$ for any $c \in B_k(p, w)$ that maximizes utility. Therefore, $\sum_N p_i s_i = \sum_{c \in K(\mu^n)} \mu(c) \sum_N p_i c_i = \sum_N p_i \sigma_i(\mu)$. Since $s_i \geq \sigma_i(\mu)$ for all i , the lemma follows. $\square$

Lemma 7. *Any CCE price is monotone.*

Proof. Suppose (p, μ) is a CCE and $\frac{p_i}{\pi_i} > \frac{p_j}{\pi_j}$ for some $s_i > s_j$. By Theorem 1, $c_i \leq c_j$ for all c with $\mu(c) > 0$. Hence, $\sigma_i(\mu) \leq \sigma_j(\mu) \leq s_j < s_i$ and thus by Lemma 6 $p_i = 0$, a contradiction with $p_j \geq 0$. $\square$

Lemma 8. *Any CCE allocation is monotone.*

Proof. Let μ be a CCE allocation and suppose that $s_i > s_j$. By Lemma 7, $\frac{p_i}{\pi_i} \leq \frac{p_j}{\pi_j}$ and by Theorem 1, $c_i \geq c_j$ for any $c \in K(\mu)$. $\square$

Lemma 9. *Any CCE allocation is measurable.*

Proof. Let μ be a CCE allocation and suppose that $s_i = s_j$. We need to show that for any $c \in K(\mu)$ we have $c_i = c_j$. Suppose toward contradiction that this is not the case and w.l.o.g. that $c_i < c_j$. By Theorem 1 this implies that $\frac{p_i}{\pi_i} > \frac{p_j}{\pi_j}$ and thus for all other $\hat{c} \in K(\mu)$ we have $\hat{c}_i \leq \hat{c}_j$. Therefore, $\sigma_i(\mu) < \sigma_j(\mu) \leq s_j = s_i$ and thus, by Lemma 6, $p_i = 0$, a contradiction with $p_j \geq 0$. $\square$

A.3 Proof of Theorem 3

Monotonicity is a consequence of Lemma 7. Uniqueness follows from the next two lemmas.

Lemma 10. *The CCE price of a pure endowment economy is unique.*

Proof. First, we show that for all c in the support of μ , $c_i > 0$ for all i . For any c in the support of μ , let $A = \{i : c_i = 0\}$. If $A \neq \emptyset$, agent optimality implies $\sum_{i \in A} p_i = 1$ and $\sum_{i \in N \setminus A} p_i = 0$; otherwise consumption can be raised by ϵ on the set A and lowered by $\frac{\epsilon \sum_{i \in A} p_i}{\sum_{i \in N \setminus A} p_i}$ on the set $N \setminus A$ resulting in an overall increase of utility for small ϵ . It follows that c costs the same as $2c$ and since $c_i > 0$ for some i and u is strictly increasing, c cannot be optimal if $A \neq \emptyset$.

Since E is a pure endowment economy, assume without loss of generality that $s_i < s_{i+1}$. For any μ , let $I(\mu) = \{i < n : c_i < c_{i+1} \text{ for some } c \in K(\mu)\}$. Since every CCE allocation solves the planner's problem and $k > 1$ (i.e., agents can have at least two distinct consumption levels), $I(\mu) \neq \emptyset$. Hence, for any competitive allocation μ , let $J(\mu) = \max I(\mu)$. Let (μ^l, p^l) for $l = 1, 2$ be two CCE.

We claim that $i \notin I(\mu^l)$ implies $i + 1 \notin I(\mu^l)$. To see why this is the case, note that if $i \notin I(\mu^l)$, then $\sigma_i(\mu^l) = \sigma_{i+1}(\mu^l)$ and therefore $\sigma_{i+1}(\mu^l) < s_{i+1}$ and hence by Lemma 6 $p_{i+1} = 0$.

Then, if $c_{i+2} > c_{i+1}$ for any $c \in K(\mu^l)$, define $\hat{c}_j = c_j$ for $j \leq i$, $\hat{c}_j = c_j$ for $j \geq i+2$ and $\hat{c}_j = c_{i+1}$ and note that $\hat{c}$ is coarse, costs the same as c but yields strictly higher utility, contradicting the fact that μ^l is a CCE allocation.

Next, we claim that $J(\mu^1) = J(\mu^2)$. If not, assume without loss of generality that $J(\mu^1) > J(\mu^2)$. Define $\hat{s}_j = s_j$ for all $j \leq J(\mu^2)$ and $\hat{s}_j = s_j + 1$ for all $j \geq J(\mu^2) + 1$. Then, since we established in the preceding paragraph that $p_j = 0$ for all $j > J(\mu^2)$, we conclude that (p^2, μ^2) is a CCE for the economy with endowment $\hat{s}$. Therefore, by Theorem 2 $W_k(s) = W_k(\hat{s})$. But, since $i := J(\mu^2) < J(\mu^1)$, the previous claim implies $i \in I(\mu^1)$. Hence, there exist $c \in K(\mu^1)$ such that $c_i < c_{i+1}$. Since c is monotone (by Theorem 1) $\hat{c}$ defined by $\hat{c}_j = c_j$ for all $j \leq i$ and $\hat{c}_j = c_j + 1$ for all $j \geq k + 1$ is coarse. Let $\hat{\mu}$ be the allocation derived from μ^1 by replacing c with $\hat{c}$. Note that $\hat{\mu}$ yields strictly higher mean utility than μ^1 and is feasible for the economy with endowment $\hat{s}$, contradicting $W_k(s) = W_k(\hat{s})$.

Note that if $J(\mu^1) = J(\mu^2) = 1$, then $p_1^1 = p_1^2 = 1$ and hence $p^1 = p^2$ as desired. So, henceforth we assume $J(\mu^1) = J(\mu^2) > 1$. By Theorem 1, both μ^1, μ^2 solve the planner's problem. Then, the linearity of W ensures that $\mu = .5\mu^1 + .5\mu^2$ also solves the planner's problem and hence by Theorem 1, there exists some p such that (p, μ) is a CCE. Then, the previous claim ensures that $J := J(\mu) = J(\mu^1) = J(\mu^2) > 1$.

For any c such that $c_j > 0$ for all j and for any $i = 1, \dots, n - 1$, define

$$MRS_i(c) = \frac{\sum_{\{j \leq i\}} \pi_j u'(c_j)}{\sum_{\{j > i\}} \pi_j u'(c_j)}$$

For the price p defined above, define $q \in \mathbb{R}^n$ such that $q_i = \sum_{j \leq i} p_j$. Define q^1, q^2 in an analogous fashion. For any $i \leq J$, pick $c^i \in K(\mu^1)$ such that $c_i^i < c_{i+1}^i$. Note that one possible coarse plan can be constructed by changing consumption in all states $j \leq i$ by ϵ and in all states $j > i$ by ϵ' in a budget neutral manner. The optimality of c^i ensures that this alternative plan cannot increase utility which means: $q_i^1 = MRS_i(c^i)(1 - q_i^1)$ for all $i \leq J$. But since $K(\mu^1) \subset K(\mu)$, the equations above also hold for q proving that $q_j = q_j^1$ for all j and hence $p^1 = p$. A symmetric argument ensures that $p^2 = p$. $\square$

Lemma 11. *The CCE price of any economy is essentially unique.*

Proof. Let $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ be any economy and let $E = (u, k, \pi, s)$ be the corresponding pure endowment economy where $\pi_i = \sum_{j: \hat{s}_j = s_i} \hat{\pi}_j$. For any measurable plan $\hat{c}$ for $\hat{E}$, define the plan

c for E in an obvious way. Then, define μ by $\mu(c) = \hat{\mu}(\hat{c})$. Given a price $\hat{p}$ for $\hat{E}$, define p for E as follows: $p_i = \sum_{j: \hat{s}_j = s_i} \hat{p}_j$ for all i .

To prove essential uniqueness, suppose there are two CCE for $\hat{E}$, $(\hat{p}^l, \hat{\mu}^l)$ for $l = 1, 2$ such that $\hat{p}^1$ and $\hat{p}^2$ are not equivalent. Then, since by Theorem 2 both $\hat{\mu}^l$ are measurable, the corresponding μ^l are well-defined allocations for E . It is easy to see that (p^l, μ^l) are CCE equilibria for E . But since $\hat{p}^1$ and $\hat{p}^2$ are not equivalent, $p^1 \neq p^2$, contradicting Lemma 10. $\square$

A.4 Dominance Lemma

Let $\{E^n\}$ be a convergent sequence of economies where $E^n = (u, k, \pi^n, s^n)$. Let (p^n, μ^n) be a CCE of E^n , let U_*^n be the equilibrium utility of an agent with wealth 1. For the remainder of this section, p^n, μ^n and U_*^n always refers to this sequence of equilibrium variables. Let $f : [a, b] \rightarrow \mathbb{R}_{++}$ be the density of the limit endowment.

Consider the problem of an agent who has wealth 1 and must choose a partition with $k - 1$ cells. For any such partition $S^n = (S_1^n, \dots, S_{k-1}^n)$, let $\pi^n(l) = \sum_{S_l^n} \pi_j^n$ and $p^n(l) = \sum_{S_l^n} p_j^n$. (For ease of notation, we write $\pi^n(l)$ instead of $\pi_{S^n}^n(l)$ and $p^n(l)$ instead of $p_{S^n}^n(l)$.)

In the following, let $\{S^n\} = \{(S_1^n, \dots, S_{k-1}^n)\}$ be a sequence of optimal partitions. By Theorem 1, each cell of S^n is contiguous, i.e., it's an interval composed of consecutive states. Let $Y^n = V(S^n)$ be the maximal utility and note that $Y^n \leq U_*^n$. We will prove the following Lemma:

Lemma 12. $\liminf_n [U_*^n - Y^n] > 0$.

The proof proceeds by contradiction. We show that if (in the limit) moving from a $(k - 1)$ -coarse plan to a k -coarse plan does not yield any utility gain, then the relative prices within each of the $k - 1$ cells are constant (Lemma 13). This implies that for any fixed cell S_l and any two cells of an optimal k -partition that nontrivially intersect S_l the relative prices on those cells must converge to each other (Lemma 14). This, in turn, implies that the aggregate demand on S_l must be constant (Lemma 15), which contradicts market clearing.

Suppose that

$$\lim [U_*^n - Y^n] = 0 \tag{a}$$

along some subsequence. Pass to that subsequence. Note that $u(a) \leq Y^n \leq U_*^n \leq u(b)$ and $0 \leq \pi^n(l) \leq 1, 0 \leq p^n(l) \leq 1$. Therefore, there exists a subsequence such that $\{Y^n, (\pi^n(l), p^n(l))_{l=1}^{k-1}\}$

converges. Pass to that subsequence and let $Y, (p(l), \pi(l))_{l=1}^{k-1}$ be its limit. By (a) we have $\lim U_*^n = Y$.

The next lemma shows that on each fragment of the set S_l^n the limit relative price is the same. For $0 < \epsilon < 1$, the set $B_l^n \subset S_l^n$ is an ϵ -fragment of S_l^n if $\sum_{B_l^n} \pi_j^n = \epsilon^n \pi^n(l)$ for some convergent sequence $\epsilon^n \rightarrow \epsilon$.

Lemma 13. *Let $\pi(l) > 0$ and, for all n , let B_l^n be an ϵ -fragment of S_l^n such that $0 < \epsilon < 1$. If $\lim U_*^n = Y$, then $\lim \sum_{B_l^n} p_j^n = \epsilon p(l)$.*

Proof of Lemma 13. Since $p_j^n \geq 0$, the conclusion of the Lemma is immediate if $p(l) = 0$. Therefore, assume $p(l) > 0$ and that B_l^n is an ϵ -fragment of S_l^n such that the conclusion of the lemma is false. Then, there is a subsequence such that $b_1^n := \sum_{B_l^n} p_j^n$ converges and

$$\left| \frac{p^n(l)}{\pi^n(l)} - \frac{b_1^n}{\epsilon^n \pi^n(l)} \right| > \delta$$

for some $\delta > 0$. Define $b_2^n := p^n(l) - b_1^n$; $a_1^n = \pi^n(l)\epsilon^n$, $a_2^n = \pi^n(l)(1 - \epsilon^n)$ and let $b_i = \lim b_i^n$, $a_i = \lim a_i^n$ for $i = 1, 2$. Then,

$$\frac{p(l)}{\pi(l)} = \frac{b_1 + b_2}{a_1 + a_2} \neq \frac{b_1}{a_1}$$

A possible plan for the agent with k -available cells is the partition

$$\hat{S}^n = (S_1^n, \dots, S_{l-1}^n, B_l^n, S_l^n \setminus B_l^n, S_{l+1}^n, \dots, S_{k-1}^n)$$

If $\sigma \neq 1$, then,

$$V(\hat{S}^n) - V(S^n) = \frac{1}{\sigma - 1} \left(b_1^n \psi_\sigma \left(\frac{a_1^n}{b_1^n} \right) + b_2^n \psi_\sigma \left(\frac{a_2^n}{b_2^n} \right) - (b_1^n + b_2^n) \psi_\sigma \left(\frac{a_1^n + a_2^n}{b_1^n + b_2^n} \right) \right)$$

If $\sigma > 1$ and $b_1, b_2 > 0$ or if $\sigma < 1$, we have

$$\lim \left(V(\hat{S}^n) - V(S^n) \right) = \frac{1}{\sigma - 1} \left(b_1 \psi_\sigma \left(\frac{a_1}{b_1} \right) + b_2 \psi_\sigma \left(\frac{a_2}{b_2} \right) - (b_1 + b_2) \psi_\sigma \left(\frac{a_1 + a_2}{b_1 + b_2} \right) \right) > 0$$

where the last inequality follows from Lemma 1. Thus, we have the desired contradiction for the case where $\sigma \neq 1$. The same argument proves the lemma for $\sigma = 1$ and $b_1, b_2 > 0$. If $b_1 = 0, b_2 > 0$ or $b_1 > 0, b_2 = 0$ and $\sigma \geq 1$ then $b_1^n \psi_\sigma \left(\frac{a_1^n}{b_1^n} \right) + b_2^n \psi_\sigma \left(\frac{a_2^n}{b_2^n} \right) \rightarrow \infty$, which proves the argument in this case. $\square$

Fix a cell l with $\pi(l) > 0$ and $p(l) > 0$. Let $0 < q_* < q_{**} < \pi(l)$. For any $j \in S_l^n$, define $L^n(j) := \{i \in S_l^n : i < j\}$ and $M^n(j) := \{i \in S_l^n : i \geq j\}$. For each n , let $\mathcal{J}^n$ be the set of states with $q_* \leq \pi^n(L^n(j)) \leq q_{**}$. Let j_*^n be the smallest member of $\mathcal{J}^n$ and let j_{**}^n be the largest. Since the distribution of the aggregate endowment F is nonatomic, it follows that $\lim_n \pi^n(L^n(j_*^n)) = q_*$ and $\lim_n \pi^n(L^n(j_{**}^n)) = q_{**}$.

Consider the cells of $\hat{S}^n$ that have a nonempty intersection with $\mathcal{J}^n$ and, of those cells, let G^n be the cell with the minimal consumption and let H^n be the cell with the maximal consumption (they may be the same cell).

Lemma 14. *Along some subsequence $\lim \frac{\pi^n(G^n)}{p^n(G^n)} = \lim \frac{\pi^n(H^n)}{p^n(H^n)}$.*

Proof of Lemma 14. Let $l_*^n \in \mathcal{J}^n$ be the maximal element of G^n and $l_{**}^n \in \mathcal{J}^n$ be the minimal element of H^n . Define $C^n := G^n \setminus L^n(l_*^n)$ and $D^n := H^n \setminus M^n(l_{**}^n)$. Let $a_1^n := \pi^n(C^n)$, $a_2^n := \pi^n(L^n(l_*^n))$, $a_3^n := \pi^n(M^n(l_{**}^n))$, $a_4^n := \pi^n(D^n)$, $b_1^n := p^n(C^n)$, $b_2^n := p^n(L^n(l_*^n))$, $b_3^n := p^n(M^n(l_{**}^n))$, $b_4^n := p^n(D^n)$. Choose a subsequence such that the limits $a_i := \lim a_i^n$ and $b_i := \lim b_i^n$ exist for $i = 1, \dots, 4$ and the ratios $\frac{\pi^n(G^n)}{p^n(G^n)}$ and $\frac{\pi^n(H^n)}{p^n(H^n)}$ converge in extended reals. Define $z := \frac{\pi(l)}{p(l)}$ and consider the following three cases:

Case 1: $\lim \frac{\pi^n(G^n)}{p^n(G^n)} < z$ and $\lim \frac{\pi^n(H^n)}{p^n(H^n)} \leq z$. Since $a_2 \geq q_* > 0$, Lemma 13 implies that $\frac{a_2}{b_2} = z$. Since G^n and $L^n(l_*^n)$ share the maximal element and are both contiguous sets, one must contain the other. If $G^n \subset L^n(l_*^n)$, then by monotonicity of the price kernel $\lim \frac{\pi^n(G^n)}{p^n(G^n)} = z$, a contradiction. Thus, $C^n \neq \emptyset$. Since H^n lies above $L^n(l_*^n)$, monotonicity of the price kernel implies that $\lim \frac{\pi^n(H^n)}{p^n(H^n)} = z$. Consider a new partition $\check{S}^n$ that instead of G^n and H^n has cells C^n and $L^n(l_*^n) \cup H^n$. Observe that $b_1 > 0$, as if $b_1 = 0$ then $\frac{a_1 + a_2}{b_1 + b_2} \geq \frac{a_2}{b_1 + b_2} = \frac{a_2}{b_2}$, a contradiction. Thus, $\frac{a_1}{b_1} < z$. For $\sigma \neq 1$ we have that

$$\begin{aligned} \lim[V(\check{S}^n) - V(\hat{S}^n)] &= \frac{1}{\sigma - 1} \left[b_1 \psi_\sigma\left(\frac{a_1}{b_1}\right) + b_2 \psi_\sigma\left(\frac{a_2}{b_2}\right) - (b_1 + b_2) \psi\left(\frac{a_1 + a_2}{b_1 + b_2}\right) \right] \\ &+ \frac{1}{\sigma - 1} \left[(b_2 + b_3 + b_4) \psi\left(\frac{a_2 + a_3 + a_4}{b_2 + b_3 + b_4}\right) - b_2 \psi\left(\frac{a_2}{b_2}\right) - (b_3 + b_4) \psi\left(\lim \frac{a_3^n + a_4^n}{b_3^n + b_4^n}\right) \right]. \end{aligned}$$

By Lemma 1, the first term is positive, while the second term is zero. The same argument proves the case $\sigma = 1$.

Case 2: $\lim_n \frac{\pi^n(G^n)}{p^n(G^n)} \geq z$ and $\lim_n \frac{\pi^n(H^n)}{p^n(H^n)} > z$. Since $a_3 \geq \pi(S_l) - q_{**} > 0$, Lemma 13 implies that $\frac{a_3}{b_3} = z$. Since H^n and $M^n(l_{**}^n)$ share the minimal element and are both contiguous

sets, one must contain the other. If $H^n \subset M^n(l_{**}^n)$, then by monotonicity of the price kernel $\lim_{p^n(H^n)} \frac{\pi^n(H^n)}{p^n(H^n)} = z$, a contradiction. Thus, $D^n \neq \emptyset$. Since G^n lies below $M^n(l_{**}^n)$, monotonicity of the price kernel implies that $\lim_{p^n(G^n)} \frac{\pi^n(G^n)}{p^n(G^n)} = z$. Consider a new partition $\bar{S}^n$ that instead of G^n and H^n has cells D^n and $M^n(l_{**}^n) \cup G^n$. Note that $b_4 = 0$ only if $\sigma < 1$. For $\sigma \neq 1$ we have

$$\begin{aligned} \lim[V(\bar{S}^n) - V(\hat{S}^n)] &= \frac{1}{\sigma - 1} \left[b_3 \psi\left(\frac{a_3}{b_3}\right) + b_4 \psi\left(\frac{a_4}{b_4}\right) - (b_3 + b_4) \psi\left(\frac{a_3 + a_4}{b_3 + b_4}\right) \right] \\ &+ \frac{1}{\sigma - 1} \left[(b_1 + b_2 + b_3) \psi\left(\frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3}\right) - (b_1 + b_2) \psi\left(\lim \frac{a_1^n + a_2^n}{b_1^n + b_2^n}\right) - b_3 \psi\left(\frac{a_3}{b_3}\right) \right]. \end{aligned}$$

By Lemma 1, the first term is positive, while the second term is zero. The same argument proves the case $\sigma = 1$.

Case 3: $\lim_n \frac{\pi^n(G^n)}{p^n(G^n)} < z$ and $\lim_n \frac{\pi^n(H^n)}{p^n(H^n)} > z$. By the same arguments as in cases 1 and 2, for n large enough $C^n \neq \emptyset$ and $D^n \neq \emptyset$. As above, $a_1, b_1, a_2, b_2, a_3, b_3, a_4 > 0$ and $b_4 = 0$ only if $\sigma < 1$. As above, we have $\frac{a_1}{b_1} < \frac{a_2}{b_2} = z = \frac{a_3}{b_3} < \frac{a_4}{b_4}$. Consider the two partitions $\check{S}^n$ and $\bar{S}^n$ from above.

Let $\lambda := \frac{b_3}{b_2 + b_3}$ and note that $a_1 + a_2 = \lambda a_1 + (1 - \lambda)(a_2 + a_3 + a_4)$, $b_1 + b_2 = \lambda b_1 + (1 - \lambda)(b_2 + b_3 + b_4)$, $a_3 + a_4 = \lambda(a_2 + a_3 + a_4) + (1 - \lambda)a_4$, and $b_3 + b_4 = \lambda(b_2 + b_3 + b_4) + (1 - \lambda)b_4$. For $\sigma \neq 1$ we have

$$\begin{aligned} V_{\check{S}} &:= \lim V(\check{S}^n) = \frac{1}{\sigma - 1} \left[b_1 \psi\left(\frac{a_1}{b_1}\right) + (b_2 + b_3 + b_4) \psi\left(\frac{a_2 + a_3 + a_4}{b_2 + b_3 + b_4}\right) \right] \\ V_{\hat{S}} &:= \lim V(\hat{S}^n) = \frac{1}{\sigma - 1} \left[(b_1 + b_2) \psi\left(\frac{a_1 + a_2}{b_1 + b_2}\right) + (b_3 + b_4) \psi\left(\frac{a_3 + a_4}{b_3 + b_4}\right) \right] \\ V_{\bar{S}} &:= \lim V(\bar{S}^n) = \frac{1}{\sigma - 1} \left[(b_1 + b_2 + b_3) \psi\left(\frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3}\right) + b_4 \psi\left(\frac{a_4}{b_4}\right) \right] \end{aligned}$$

By Lemma 1 $\frac{1}{\sigma - 1} [(b_1 + b_2) \psi(\frac{a_1 + a_2}{b_1 + b_2})] \leq \frac{1}{\sigma - 1} [\lambda b_1 \psi(\frac{a_1}{b_1}) + (1 - \lambda)(b_1 + b_2 + b_3) \psi(\frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3})]$ and $\frac{1}{\sigma - 1} [(b_3 + b_4) \psi(\frac{a_3 + a_4}{b_3 + b_4})] \leq \lambda(b_2 + b_3 + b_4) \psi(\frac{a_2 + a_3 + a_4}{b_2 + b_3 + b_4}) + (1 - \lambda)b_4 \psi(\frac{a_4}{b_4})$, so $V_{\check{S}} \leq \lambda V_{\hat{S}} + (1 - \lambda)V_{\bar{S}}$. Moreover, equality holds only if $\frac{a_1}{b_1} = \frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3}$, which is inconsistent with the assumption. Thus, $V_{\check{S}} < \lambda V_{\hat{S}} + (1 - \lambda)V_{\bar{S}} \leq \max\{V_{\check{S}}, V_{\bar{S}}\}$. The same argument proves the case $\sigma = 1$.

Since cases 1-3 were ruled out, it follows that $\lim_n \frac{\pi^n(G^n)}{p^n(G^n)} \geq z$ and $\lim_n \frac{\pi^n(H^n)}{p^n(H^n)} \leq z$. Then by monotonicity of the price kernel they are both equal to z as desired. $\square$

Lemma 15. $\lim_{n \rightarrow \infty} [\sigma_{j_{**}}^n(\mu^n) - \sigma_{j_*}^n(\mu^n)] = 0$.

Proof of Lemma 15. Suppose toward contradiction that there exists a convergent subse-

quence where $\lim[\sigma_{j_{**}^n}^n(\mu^n) - \sigma_{j_*^n}^n(\mu^n)] > 0$. Thus, there is a sequence $\hat{S}^n$ of optimal partitions for which the difference between the consumption in states j_{**}^n and j_*^n is bounded away from zero for large n . This implies that the sets G^n and H^n defined above are distinct, so by Lemma 14, $\lim \frac{\pi^n(G^n)}{p^n(G^n)} = \lim \frac{\pi^n(H^n)}{p^n(H^n)}$. Let x^n and y^n be the optimal consumption levels on cells G^n and H^n . Then, consumption in state j_*^n is x^n and consumption in state j_{**}^n is y^n . With CRRA utility, we have $x^n = \left(\frac{\pi^n(G^n)}{p^n(G^n)}\right)^\sigma w^n$ and $y^n = \left(\frac{\pi^n(H^n)}{p^n(H^n)}\right)^\sigma w^n$, where

$$\frac{1}{w^n} = \sum_{l' \neq l} (\bar{p}_{l'}^n)^\sigma (\beta_{l'}^n)^{1-\sigma} + (p^n(G^n))^\sigma (\pi^n(G^n))^{1-\sigma} + (p^n(H^n))^\sigma (\pi^n(H^n))^{1-\sigma}.$$

Thus, $|y^n - x^n| \leq w^n \left[\left(\frac{\pi^n(H^n)}{p^n(H^n)}\right)^\sigma - \left(\frac{\pi^n(G^n)}{p^n(G^n)}\right)^\sigma \right]$. Since, $w^n \leq \frac{z}{q_*}$, we have that $|y^n - x^n| \rightarrow 0$, a contradiction. $\square$

Since the density of the aggregate endowment f is continuous on $[a, b]$, it is bounded, which implies that $\lambda := \lim_n [s_{j_{**}^n}^n - s_{j_*^n}^n] > 0$. Lemma 6 then implies that $p_{j_*^n}^n = 0$ for large n , which, in turn, implies that $p^n(M^n(j^*)) = 0$, which by Lemma 13 is a contradiction with $p(l) > 0$ and $\pi^n(M^n(j^*)) > \pi(l) - q_{**} > 0$, thus completing the proof of the dominance lemma.

A.5 Proof of Theorem 4

Let $\{E^n\}$ be a convergent sequence of economies where $E^n = \{(u, k, \pi^n, s^n)\}$. Let $\{p^n\}$ be the CCE price of $\{E^n\}$. In the following, $S^n = (S_1^n, \dots, S_k^n)$ is a sequence of optimal partitions. We write $j^n(l)$ to denote the maximal element of S_l^n . Hence, $S_l^n = \{j^n(l-1) + 1, \dots, j^n(l)\}$ with $j^n(0) = 0$. The sequence $\{(\pi^n(l), p^n(l))_{l=1}^k\}$ is the corresponding sequence of prices and probabilities of the k cells, that is, $\pi^n(l) = \sum_{S_l^n} \pi_j^n$ and $p^n(l) = \sum_{S_l^n} p_j^n$. We consider a subsequence along which $\{(\pi^n(l), p^n(l))_{l=1}^k\}$ converges and write $\{(\pi(l), p(l))_{l=1}^k\}$ for its limit.

In the proofs below, we repeatedly use the fact that $\pi(m)p(m) > 0$ for some $m \in \{1, \dots, k\}$. To see why this is true, note that if $\sum_{l=1}^k \pi(l)p(l) = 0$ then $V(S^n)$ converges to its upper bound ($+\infty$ if $\sigma \geq 1$ and 0 if $\sigma < 1$) which is inconsistent with feasibility. We also use the fact that $p_{j+1}^n > 0$ implies that there exists an optimal plan such that $j^n(l) = j$. To prove this assertion, assume it were false. Then, the aggregate demand in state j is equal to the aggregate demand in state $j+1$ and, therefore, $p_{j+1} = 0$ by Lemma 6.

Lemma 16. Suppose $\pi^n(l) \rightarrow 0$. (1) If $\sigma < 1$, then there exists $\delta > 0$ such that $p^n(l) \geq \delta$. (2) If $\sigma \geq 1$, then there exists $\delta > 0$ such that either $p^n(l) \geq \delta$ or $\frac{p^n(l)}{\pi^n(l)} \rightarrow 0$.

Proof. Let $m \in \{1, \dots, k\}$ be such that $p(m)\pi(m) > 0$. Let $\hat{S}^n$ be a partition that combines cells S_m^n and S_l^n . For $\sigma \neq 1$ we have

$$\begin{aligned} (\sigma - 1)(V(S^n) - V(\hat{S}^n)) &= p^n(m)\psi_\sigma\left(\frac{\pi^n(m)}{p^n(m)}\right) + p^n(l)\psi_\sigma\left(\frac{\pi^n(l)}{p^n(l)}\right) \\ &\quad - (p^n(m) + p^n(l))\psi_\sigma\left(\frac{\pi^n(m) + \pi^n(l)}{p^n(m) + p^n(l)}\right) \end{aligned}$$

and for $\sigma = 1$ we have

$$\begin{aligned} V(S^n) - V(\hat{S}^n) &= p^n(m)\psi_\sigma\left(\frac{\pi^n(m)}{p^n(m)}\right) + p^n(l)\psi_\sigma\left(\frac{\pi^n(l)}{p^n(l)}\right) \\ &\quad - (p^n(m) + p^n(l))\psi_\sigma\left(\frac{\pi^n(m) + \pi^n(l)}{p^n(m) + p^n(l)}\right) \end{aligned}$$

Assume that the conclusion of the Lemma is false. If $\sigma < 1$ then this implies that $p^n(l) \rightarrow 0$ and, therefore, $p^n(l)\psi_\sigma\left(\frac{\pi^n(l)}{p^n(l)}\right) \rightarrow 0$. If $\sigma \geq 1$ then this implies that $p^n(l) \rightarrow 0$ and $\frac{p^n(l)}{\pi^n(l)} \geq \delta$ for some $\delta > 0$. Again, it follows that $p^n(l)\psi_\sigma\left(\frac{\pi^n(l)}{p^n(l)}\right) \rightarrow 0$. Since $p(m)\pi(m) > 0$ and ψ is continuous, it follows that $\lim \psi\left(\frac{\pi^n(m) + \pi^n(l)}{p^n(m) + p^n(l)}\right) = \psi\left(\frac{\pi(m)}{p(m)}\right)$ and we conclude $V(S^n) - V(\hat{S}^n) \rightarrow 0$ contradicting Lemma 12. $\square$

Lemma 17. Assume $\sigma \leq 1$ and $\lim s_{j^n(l)}^n = a$ for some $l = 1, \dots, k$. Then, $\lim \frac{p_{j^n(l)}^n}{\pi_{j^n(l)}^n} = \infty$.

Proof. Since s^n converges in distribution to a random variable with a continuous density, the hypothesis of the Lemma implies that $\sum_{l' \leq l} \pi^n(l') \rightarrow 0$ and, in particular, $\pi^n(l) \rightarrow 0$. If $\sigma < 1$, this and Lemma 16 imply that $p(l) \geq \delta$ for some $\delta > 0$. If $\sigma = 1$, this and Lemma 16 imply that either $p(l) \geq \delta$ or $\frac{p^n(l)}{\pi^n(l)} \rightarrow 0$. If the latter is true, then Theorem 3 implies that $p(l')\pi(l') = 0$ for all $l' > l$. Since $\sum_{l'=l+1}^k \pi(l') = 1$, this is impossible. Therefore, $p(l) \geq \delta$ also in this case.

Assume, contrary to the assertion of the Lemma, that $\frac{p_{j^n(l)}^n}{\pi_{j^n(l)}^n} \leq K < \infty$ along some subsequence. First, consider the case where $\sigma < 1$. Let m be such that $p(m)\pi(m) > 0$ and consider the partition $\hat{S}^n$ identical to S^n except that the state $j^n(l)$ is moved from S_l^n to S_m^n . Then,

$V(S^n) - V(\hat{S}^n) = \frac{1}{1-\sigma}(f^n - g^n)$ where

$$\begin{aligned} f^n &:= p^n(l)^{1-\sigma} \pi^n(l)^\sigma - (p^n(l) - p_{j^n(l)}^n)^{1-\sigma} (\pi^n(l) - \pi_{j^n(l)}^n)^\sigma \\ g^n &:= (p^n(m) + p_{j^n(l)}^n)^{1-\sigma} (\pi^n(m) + \pi_{j^n(l)}^n)^\sigma - p^n(m)^{1-\sigma} \pi^n(m)^\sigma \end{aligned}$$

Note that by concavity of the power function

$$\frac{f^n}{\pi_{j^n(l)}^n} \geq \frac{p^n(l)^{1-\sigma}}{\pi_{j^n(l)}^n} (\pi^n(l)^\sigma - (\pi^n(l) - \pi_{j^n(l)}^n)^\sigma) \geq \frac{p^n(l)^{1-\sigma}}{\pi_{j^n(l)}^n} \left(\sigma \frac{\pi_{j^n(l)}^n}{\pi^n(l)^{1-\sigma}} \right) = \sigma \left(\frac{p^n(l)}{\pi^n(l)} \right)^{1-\sigma}$$

Since $\lim \pi^n(l) = 0$ and $\lim p^n(l) = p(l) > 0$ it follows that $\frac{f^n}{\pi_{j^n(l)}^n} \rightarrow \infty$. Furthermore, since $p_{j^n(l)}^n \leq K \pi_{j^n(l)}^n$,

$$\frac{g^n}{\pi_{j^n(l)}^n} \leq \frac{1}{\pi_{j^n(l)}^n} \left((p^n(m) + K \pi_{j^n(l)}^n)^{1-\sigma} (\pi^n(m) + \pi_{j^n(l)}^n)^\sigma - p^n(m)^{1-\sigma} \pi^n(m)^\sigma \right)$$

Since $p(m) > 0$, the right hand side of this inequality converges to $(1-\sigma)K \left(\frac{\pi(m)}{p(m)} \right)^\sigma + \sigma \left(\frac{p(m)}{\pi(m)} \right)^{1-\sigma}$. Thus, for large n it follows that $f^n - g^n > 0$ and, therefore, $V(S^n) < V(\hat{S}^n)$. This contradicts the optimality of S^n .

Next, consider the case $\sigma = 1$. Then, $V(S^n) - V(\hat{S}^n) = \tilde{f}^n - \tilde{g}^n$ where

$$\begin{aligned} \tilde{f}^n &:= \pi^n(m) \ln \left(\frac{\pi^n(m)}{p^n(m)} \right) - (\pi^n(m) + \pi_{j^n(l)}^n) \ln \left(\frac{\pi^n(m) + \pi_{j^n(l)}^n}{p^n(m) + p_{j^n(l)}^n} \right) \\ \tilde{g}^n &= (\pi^n(l) - \pi_{j^n(l)}^n) \ln \left(\frac{\pi^n(l) - \pi_{j^n(l)}^n}{p^n(l) - p_{j^n(l)}^n} \right) - \pi^n(l) \ln \left(\frac{\pi^n(l)}{p^n(l)} \right) \end{aligned}$$

if the cell l contains elements other than j^n , and otherwise $\tilde{g}^n = -\pi^n(l) \ln \left(\frac{\pi^n(l)}{p^n(l)} \right)$. In the latter case, $\frac{\tilde{g}^n}{\pi_{j^n(l)}^n} = -\ln \left(\frac{\pi^n(l)}{p^n(l)} \right) \rightarrow \infty$ since $\lim \pi^n(l) = \lim \pi_{j^n(l)}^n = 0$. In the former case, since $\pi^n(l) \geq \pi_{j^n(l)}^n$ it follows from the convexity of the function $t \mapsto t \ln(t/p^n(l))$ that

$$\frac{\tilde{g}^n}{\pi_{j^n(l)}^n} \geq \frac{(\pi^n(l) - \pi_{j^n(l)}^n) \ln \left(\frac{\pi^n(l) - \pi_{j^n(l)}^n}{p^n(l)} \right) - \pi^n(l) \ln \left(\frac{\pi^n(l)}{p^n(l)} \right)}{\pi_{j^n(l)}^n} \geq \frac{\pi_{j^n(l)}^n (-1 - \ln(\frac{\pi^n(l)}{p^n(l)}))}{\pi_{j^n(l)}^n} \rightarrow \infty.$$

Furthermore,

$$\frac{\tilde{f}^n}{\pi_{j^n(l)}^n} \leq \frac{\pi^n(m) \ln \left(\frac{\pi^n(m)}{p^n(m)} \right) - \left(\pi^n(m) + \pi_{j^n(l)}^n \right) \ln \left(\frac{\pi^n(m) + \pi_{j^n(l)}^n}{p^n(m) + K \pi_{j^n(l)}^n} \right)}{\pi_{j^n(l)}^n}$$

Since $p(m) > 0$, the right hand side of the above inequality converges to $K \frac{\pi(m)}{p(m)} - \ln \frac{\pi(m)}{p(m)} - 1$. It follows that $\tilde{f}^n < \tilde{g}^n$ for large n and, therefore, $V(S^n) < V(\hat{S}^n)$ for large n , contradicting the optimality of S^n . $\square$

Proof of Theorem 4: Let ι^n be such that $s_{\iota^n+1} \rightarrow a$. Then, for n sufficiently large there exists an optimal plan such that $j^n(l) = \iota^n$. To prove this assertion, assume it were false along a sequence of equilibria. Then, $p_{\iota^n+1} = 0$ and, by Theorem 3, $p_{\iota^n+k} = 0$ for all $k \geq 2$. But this implies that $\sum_{l=1}^k \pi(l)p(l) = 0$, a contradiction.

Note that $\sum_{l' > l} \pi(l') = 1$ and, therefore, $\sum_{l' > l} p(l')\pi(l') > 0$. Since normalized prices are monotone, it follows that $0 < \sum_{l' > l} p^n(l')\pi^n(l') \leq \frac{p^n(l)}{\pi^n(l)} \sum_{l' > l} \pi^n(l')\pi^n(l')$ and, therefore, $\frac{p^n(l)}{\pi^n(l)}$ must stay bounded away from zero. Therefore, Lemma 16 implies that $\sum_{j=1}^{\iota^n} p_j^n \geq \delta$ for some $\delta > 0$. Since $\iota^n = 1$ is an example of a sequence that satisfies $s_{\iota^n+1} \rightarrow a$ this, in turn, implies that $p_1^n \geq \delta$ for all n . This proves the first assertion of part (i) of the theorem. Moreover, when $\sigma \leq 1$, Lemma 17 implies that $\frac{p_{\iota^n}}{\pi_{\iota^n}} \rightarrow \infty$, which proves the second assertion of part (i) of the theorem.

Theorem 3 and the fact that $\sum_{i \in N} p_i^n = 1$ imply that

$$\sum_{l'=l}^k \pi^n(l') \geq \sum_{l'=l}^k p^n(l'). \quad (*)$$

If $\sigma < 1$, let ι^n be the largest $i \in N$ such that $p_{\iota^n+1}^n > 0$. It follows that there is an optimal plan such that $j^n(l-1) = \iota^n$ for some $l \leq k$. Therefore, (*) together with Lemma 16 part (1) imply that $\sum_{l'=l}^k \pi^n(l')$ must stay bounded away from zero. This proves the second assertion of part (ii) of the theorem, which also implies the first assertion in the case of $\sigma < 1$.

To complete the proof of the first assertion of part (ii) of the theorem in the case of $\sigma \geq 1$, let $\iota^n < n$ be such that $s_{\iota^n+1} \rightarrow b$. Since $p_{\iota^n+1}^n > 0$, it follows that there is an optimal plan such that $j^n(l-1) = \iota^n$ for some $l \leq k$. Since $\sum_{l'=l}^n \pi^n(l') \rightarrow 0$ it follows from (*), Lemma 16, and Theorem 3 that $\frac{\sum_{l'=l}^k p^n(l')}{\sum_{l'=l}^k \pi^n(l')} \rightarrow 0$. Suppose, contrary to the assertion of the Theorem, that

there is $\epsilon > 0$ and j^n such that $s_{j^n} \rightarrow b$ and $\frac{p_{j^n}^n}{\pi_{j^n}^n} \geq \epsilon$ for all n . Let $\beta^n := \sum_{i \geq j^n} \pi_i^n$ and let $i^n := \max\{i \in N : \sum_{j \geq i} \pi_j^n \geq 2\beta^n\}$. Note that $\sum_{i \geq i^n} \pi_i^n < 2\beta^n + \pi_{i^n}^n$. Since $\pi_{i^n}^n \rightarrow 0$, it follows from the argument above that $\sum_{i \geq i^n} \pi_i^n \rightarrow 0$. Therefore, $\{i^n\}$ satisfies the conditions of claim above and $\lim \sum_{i \geq i^n} p_i^n / \sum_{i \geq i^n} \pi_i^n = 0$. But $p_i^n / \pi_i^n \geq \epsilon$ for all $i^n \leq i \leq j^n$ and therefore

$$\frac{\sum_{i \geq i^n} p_i^n}{\sum_{i \geq i^n} \pi_i^n} \geq \frac{\epsilon \beta^n}{2\beta^n + \pi_{i^n}^n} \rightarrow \frac{\epsilon}{2} > 0$$

yielding the desired contradiction. This proves the first assertion of part (ii) of the theorem. $\square$

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